Rational and Real Numbers
Activity 9
Math-T101 Spring 2014

Name: Serife

(*) decimals: a new way of writing fractions

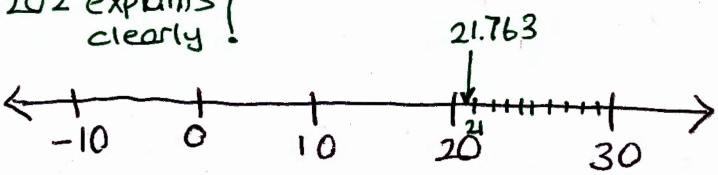
1 Decimal expansion of numbers

Problem 1. *Write down the expanded form for* 21.763 *and place it on the number line.*

$$21.763 = (2\times 10) + (1\times 1) + \left(7\times \frac{1}{10}\right) + \left(6\times \frac{1}{100}\right) + \left(3\times \frac{1}{1000}\right)$$

$$= (2\times 10) + (1\times 1) + (7\times 0.1) + (6\times 0.01) + (3\times 0.001)$$

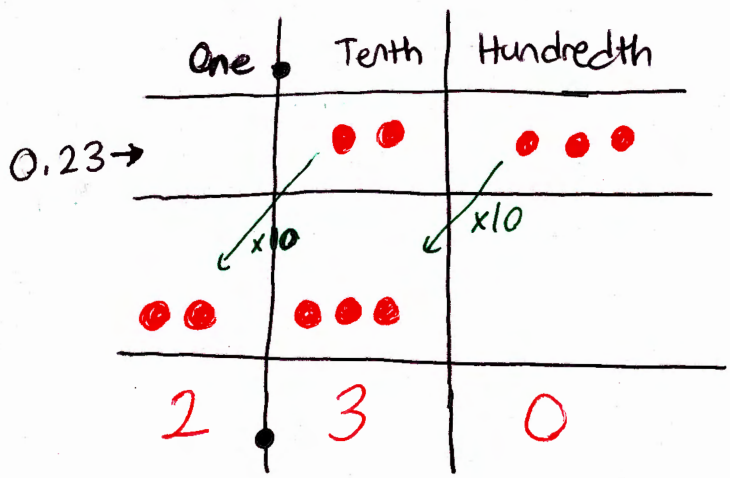
* p.202 explains clearly!



Tens	ones		Tenths	Hundredths	Thousandths
2	1	.	7	6	3

2 Multiplying and Dividing by Ten

Problem 2. *Use a chip model to find* 0.23×10.



$0.23 \times 10 = 2.30$

(*) Addition, Subtraction } Is similar to the chip model in whole numbers
↓
involves grouping, ungrouping or composing, decomposing

(*) $0.23 \xrightarrow{\times 10} 2.3$ $\qquad$ $0.23 \xrightarrow{\div 10} 0.023$

3 Multi-digit Decimal Multiplication and Division

Problem 3. *Find* 1.02 × 1.5 *in decimal form by writing*

$$1.02 \times 1.5 = \frac{102}{100} \times \frac{15}{10} = \frac{102.15}{1000} = \frac{180}{1000} = 0.180 = 0.18$$

Use this to justify the SCA for 1.02 × 1.5. *Show the SCA for this example.*

(*) multiply by ignoring the decimal point, Then add the decimal point for the sum of the digits after decimal point of each number [i.e. the sum of # of decimal places]

$$\frac{102.15}{1000} = 0.180$$

multiplication by ignoring decimal point

consider the sum of digits after decimal points.

Problem 4. *Find* 0.08888 ÷ 2.2 *in decimal form by writing*

$$0.08888 \div 2.2 = \frac{8888}{100000} \div \frac{22}{10} = \frac{8888 \div 22}{100000 \div 10} = \frac{404}{10000} = 0.0404$$

Use this to justify the SCA for 0.08888 ÷ 2.2. *Show the SCA for this example.*

$$2.2\overline{)0.08888} \rightarrow 22\overline{)0.8888}$$

$$\begin{array}{r} 0.0404 \\ 22\overline{)0.8888} \\ -88 \\ \hline 088 \\ -88 \\ \hline 0 \end{array}$$

$$\frac{0.08888 \times 10}{2.2 \times 10} = \frac{0.8888}{22}$$

(*) moving decimal point one digit means multiplying by 10.

OR

$$2.20000\overline{)0.08888} \rightarrow 220000\overline{)8888}$$

0.0404

General Procedure: shift the decimal point of the divisor to make it whole number
Shift the decimal point of dividend the same number places
Find the quotient & align the decimal point of the quotient & dividend

4 Rational Numbers and Decimals

Definition 1. *Rational numbers are numbers that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.*

Note: Every whole number is an integer (but not vice versa); every integer is a rational number (but not vice versa); and every rational number is a real number (but not vice versa).

Problem 5. *Write the following numbers in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.*

1. $1.25 = \frac{125}{100} = 1\frac{25}{100} = 1\frac{1}{4}$ (R)
2. $3 = \frac{3}{1}$ (R)
3. $-5 = -\frac{5}{1}$ (R)
4. $0.317826 = \frac{317826}{1000\,000}$ (R)

e.g. 2.2 ⇒ may not look like a fraction but it can be written as $\frac{22}{10}$, so it is rational number.

Fact 1. *A rational number, given as $\frac{a}{b}$ in simplest (reduced) form, can be written as a finite decimal if and only if the denominator b can be written as $b = 2^n \times 5^m$, where n and m are whole numbers. Note that we allow n and/or m to be 0; in particular, it is possible for b to be 1.*

Problem 6. *Write $\frac{3}{160}$ as a finite decimal.*

→ terminates if the fraction's denominator is a ~~multiple~~ power of 10 (2x5).

$$\frac{3}{160} = \frac{3}{2^4 \cdot 10} = \frac{3}{2^5 \cdot 5} \cdot \frac{5^4}{5^4} = \frac{3 \cdot 5^4}{2^5 \cdot 5^5} = \frac{3 \cdot 5^4}{10^5}$$

$$\begin{array}{r} 0.01875 \\ 160 \overline{)\,300} \\ -160 \\ \hline 1400 \\ -1280 \\ \hline 1200 \\ -1120 \\ \hline 800 \\ -800 \\ \hline 000 \end{array}$$

→ terminated after 5th digit

5 Rational numbers with infinite repeating decimal expansions

Problem 7. *Use long division to write $\frac{1}{7}$ as a repeating decimal.*

$$7 \overline{)\,10} \quad 0.\overbrace{142857}^{\text{repeats}}142857 1\ldots$$

$\frac{1}{7} = 0.\underline{\qquad}$

repeats after max. 6 digits

period : # of repeating digits

e.g.: $\frac{1}{17}$ → repeats after max. 16 digits. | $\frac{1}{13}$ → repeats after 6 digits not 12.

Problem 8. *Write $0.\overline{57}$ as a fraction.*

$$x = 0.\overline{57} \xrightarrow{\times 100} 100x = 57.\overline{57}$$

multiply by 100 because 2 digits repeating

$$100x - x = 57.\overline{57} - 0.\overline{57}$$

$$99x = 57$$

$$x = \frac{57}{99}$$

OR

$$\begin{array}{rl} 100x &= 57.\overline{57} \\ - \quad x &= 0.\overline{57} \\ \hline 99x &= 57 \end{array}$$

Problem 9. *Write $2.3\overline{715}$ as a fraction.*

$$x = 2.3\overline{715} \xrightarrow{\times 10} 10x = 23.\overline{715}$$

→ first get the repeating decimals just after decimal point

Then use the method above.

$$10x = 23.\overline{715} \xrightarrow{\times 1000} 10000x = 23715.\overline{715}$$

$$10000x - 10x = 23715.\overline{715} - 23.\overline{715}$$

$$9990x = 23715 - 23$$

$$9990x = 23692$$

$$x = \frac{23692}{9990}$$

Is $0.\bar{9} = 1$? $\longrightarrow$ YES!

e.g $6.\bar{9} = 7$

↑ exactly! not approximately.

$$\left.\begin{aligned} x &= 0.\bar{9} \\ 10x &= 9.\bar{9} \end{aligned}\right\} \quad (\times 10)$$

$$10x - x = 9.\bar{9} - 0.\bar{9}$$
$$9x = 9 - 0$$
$$x = \frac{9}{9} = 1$$

* every repeating decimal is a rational number

Definition: A real number is a number that can be written in decimal form as $\pm a.b_1b_2b_3\cdots$, where a is a whole number and each b_i is an element of $\{0, 1, 2, \cdots, 9\}$.

Fact: Every number on the number line is a real number, either rational or irrational.

Problem 10. *Give 5 examples of different irrational numbers.*

$\sqrt{2}$, π, e

$0.12317813\ldots$

$0.101001000100001\ldots$

Problem 11. *Prove that sum of two rational numbers is still a rational number.*

Assume $\frac{a}{b}$ & $\frac{c}{d}$ are two rational numbers

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\longrightarrow$ in the fraction form, so it is rational.

Problem 12. *Is the result in the preceding problem true if "rational" is replaced by "irrational"? That is, is the sum of two irrational numbers always an irrational number? If yes, explain why, and if no, give a counterexample.* $\longrightarrow$ NO

$\underbrace{\pi}_{\text{irrational}} + \underbrace{(-\pi)}_{\text{irrational}} = \underbrace{0}_{\text{rational}}$

what if, adding two positive irrational #s?

$\sqrt{2} + \sqrt{2} = \underbrace{2\sqrt{2}}_{\text{irrational}}$,

$\underbrace{0.1010010001\ldots}_{\text{irrational}} + \underbrace{0.010110111 0\ldots}_{\text{irrational}} = \underbrace{0.\bar{1}}_{\text{rational because } \frac{1}{9}}$

Problem 13. *(Do at home for next class.) Arrange the following real numbers in increasing order and write "=" for any two that are the same:* $0.98, 0.\overline{98}, 0.9\overline{8}, 0.8\overline{98}, 0.8\overline{9}, 0.9,$ $0.890, 0.90, 0.9988, 0.9899, 0.989889888988889\cdots, 0.9889888988889\cdots, 0.8\overline{99}, -99.8\overline{9}, -99.889$

$0.9800000 = 0.98$

$0.\overline{98} = 0.989898\ldots$

$0.9\overline{8} = 0.98888\ldots$

$0.8\overline{98} = 0.89898\ldots$

$0.8\overline{9} = 0.8999\ldots$

$0.9 = 0.900000$

$0.890 = 0.89000$

$0.90 = 0.90000$

$0.9988 = 0.998800$ (biggest)

$0.9899 = 0.989900$

$0.8\overline{99} = 0.89999$

$-99.8\overline{9} = -99.8999$ (smallest)

$-99.889 = -99.8890$

Problem 14. *Find a rational and an irrational number between* $0.2000020000002000000002\cdots$ *and* $0.2000200002000002000000002\cdots$.

$0.20000200\ldots$

$0.2000100001000001\ldots$ → irrational OR 0.20001 → rational

$0.20002000\ldots$

6 Preliminary Fact

Recall from Lemma 4 in Class Activity 5, and Lemma 5.14 on p.129 of Parker and Baldridge:

Fact 2. *If p is a prime factor of ab, where a and b are whole numbers, then p is a factor of either a or b.*

This fact will be used in the proofs given below that $\sqrt{2}$ and $\sqrt{7}$ are irrational.

7 Proof by Contradiction

In order to prove a statement S by contradiction, you suppose that S is false, and you show that this leads to a contraction.

Example 1. *Here is an example of a (non-mathematical) proof by contradiction. Let S be the following statement: A suspect who has a solid alibi placing him in downtown Chicago at 5:30 PM on July 1, 2013, could not have committed a hold-up at a bank in downtown Bloomington, IN at 5:00 PM on the same day.*

Proof of statement S: *Suppose he had committed the hold-up. Then he would have traveled from downtown Bloomington, IN to downtown Chicago in half an hour. This is not possible. (The fastest way would be to go by private plane from the Bloomington airport, but it takes 15 minutes to get to this airport from downtown, and it takes more than 15 minutes for the flight.) Therefore he didn't commit this crime.*

8 Proof that $\sqrt{2}$ is Irrational

Proof: Suppose $\sqrt{2}$ were rational. Then we can write $\sqrt{2} = \frac{a}{b}$, where a and b are positive whole numbers, and the fraction $\frac{a}{b}$ is in simplest form, that is, a and b are relatively prime. By squaring both sides of the equation $\sqrt{2} = \frac{a}{b}$, we obtain $2 = \frac{a^2}{b^2}$. This implies that $a^2 =$ $2b^2$. So a^2 is even (even or odd?). What does this tell you about a? Could a be odd?

because it is a multiple of 2

- a is even k has to be even.

- no

So, $a = 2k$ for some k → Then $a^2 = 2b^2$, $a^2 = 4k^2$ } $2b^2 = 4k^2$, $b^2 = 2k^2$ → b^2 is even, so b

$a^2 = 4k^2$

contradiction because a & b was assumed to be in their simplest form with having no common factor

If we write $a = 2k$, what can you conclude about b^2? (Is is even or odd?) What about b?

b^2 is even

so b is, because even · any number = even

Why is this a contradiction? (What did we assume about a and b?)

9 Proof that $\sqrt{7}$ is Irrational

Proof: This proof is similar to the proof that $\sqrt{2}$ is irrational, but we replace "even" by "divisible by 7" and we replace "odd" by "not divisible by 7." Suppose that $\sqrt{7}$ were rational. Then we can write $\sqrt{7} = \frac{a}{b}$, where a and b are positive whole numbers, and the fraction $\frac{a}{b}$ is in simplest form, that is, a and b are relatively prime. By squaring both sides of the equation we obtain ...

$$(\sqrt{7})^2 = \left(\frac{a}{b}\right)^2 \rightarrow 7 = \frac{a^2}{b^2} \rightarrow a^2 = 7b^2$$

This implies that a^2 is div. by 7 (divisible by 7 or not divisible by 7?) What does this tell you about a? (Is it divisible by 7 or not? Hint: Make use of Fact 2.)

If a^2 is div. by 7, then either a is div. by 7 or a is div. by 7 (Fact 2)

a.a

, so a has to be div. by 7.

Finish the argument by considering the divisibility of b^2 and b by 7. What is the contradiction?

Then $a = 7k$, $a^2 = 49k^2$ → so $a^2 = 7b^2$, $a^2 = 49k^2$ } $7b^2 = 49k^2$, $b^2 = 7k^2$

So b^2 is div. by 7, which implies that b is div. by 7 (Fact 2)

Contradiction

Contradiction because a & b were assumed to be in their simplest forms with having no commond factor, however they appeared to have 7 as common factor.