Orthogonal Projections
Lay 6.3

1 A Decomposition

Consider a vector $\mathbf{y} \in \mathbb{R}^n$, and suppose we are given some subspace W of $\mathbb{R}^n$. We want to decompose $\mathbf{y}$ into a part lying in W and a part not in W. That is, write $\mathbf{y}$ as a linear combination of two vectors, one lying in W and one not in W. But there are many ways to do this, as the following example shows:

Example 1.1. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and let $W = \text{span}\{\mathbf{e}_1\}$. Then $\mathbf{y} = \mathbf{e}_1 + \mathbf{e}_2$ is one such decomposition of $\mathbf{y}$. But $2\mathbf{e}_1 + (-\mathbf{e}_1 + \mathbf{e}_2)$ is another (since $-\mathbf{e}_1 + \mathbf{e}_2$ is not in W).

So there are potentially many ways to construct such decompositions. However, if we impose a stricter condition on the second vector than "not in W," we can define a *unique* decomposition which we will call "orthogonal decomposition" which has a number of nice uses.

2 Orthogonal projection

The trick is to ask for a decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^\perp$, the orthogonal complement of W. We call this an **orthogonal decomposition** of $\mathbf{y}$, and we call $\hat{\mathbf{y}}$ the **orthogonal projection of $\mathbf{y}$ onto W**.

Theorem 2.1. *Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written uniquely as*

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z},$$

where $\hat{\mathbf{y}} \in W$ *and* $\mathbf{z} \in W^{\perp}$. *Moreover, if we know an orthogonal basis* $\mathbf{u}_1, \ldots, \mathbf{u}_p$ *for* W*, then we can find the orthogonal projection* $\hat{\mathbf{y}}$ *by*

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \ldots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p.$$

Note that the formula for the orthogonal projection above looks similar to the one for basis coefficients in terms of an orthogonal basis.

Example 2.2. Consider $\mathbb{R}^3$, with $W = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, and let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthogonal basis for W, we can use the above formula to see that $\hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. Recall the distance function dist, where $\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. Notice that $\text{dist}(\mathbf{y}, \hat{\mathbf{y}}) = \sqrt{4^2} = 4$.

On the other hand, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ is another vector in W, then $\text{dist}(\mathbf{y}, \mathbf{v}) = \sqrt{(2 - v_1)^2 + (3 - v_2)^2 + 4^2}$, which is > 4 if $\mathbf{v} \neq \hat{\mathbf{y}}$. So $\hat{\mathbf{y}}$ is the closest vector in W to $\mathbf{y}$. We will see in what follows that this is not a coincidence.

Note that $\hat{\mathbf{y}}$ is independent of the choice of basis for W, as long as the basis is orthogonal (you can try this in our specific example by checking what you get if you use the orthogonal basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

We close this section by noting that we will often prefer the notation $\text{proj}_W \mathbf{y}$ for $\hat{\mathbf{y}}$, since it makes clear exactly what subspace we are projecting onto.

3 Properties of the orthogonal projection

One important property of the orthogonal projection:

Theorem 3.1. *If* $\mathbf{y} \in W$*, then* $\text{proj}_W \mathbf{y} = \mathbf{y}$.

Another is the following, which says that our "closest vector" observation from the past example is a general fact:

Theorem 3.2. *If W is a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$, then* $\text{proj}_W y$ *is the closest vector in W to $\mathbf{y}$. What we mean by this is that*

$$\text{dist}(\mathbf{v}, \mathbf{y}) > \text{dist}(\text{proj}_W \mathbf{y}, \mathbf{y})$$

for any $\mathbf{v} \in W$ such that $\mathbf{v} \neq \text{proj}_W \mathbf{y}$.

The reason the above theorem holds is because of the Pythagorean theorem for orthogonal vectors; see Figure 4 in Lay for a good picture of how the orthogonal projection looks, which will help you understand why it is true.

4 A cute representation

If the basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is orthonormal, then the formula for $\text{proj}_W \mathbf{y}$ is simpler, since we can drop all factors that look like $\mathbf{u}_i \cdot \mathbf{u}_i$ (since $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for all i). This fact allows for the following cute representation of the orthogonal projection:

Theorem 4.1. *If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W (a subspace of $\mathbb{R}^n$), then, defining*

$$U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_p \end{bmatrix},$$

we have

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y}$$

for all $\mathbf{y} \in \mathbb{R}^n$.

This is just a consequence of the definitions (writing out $UU^T\mathbf{y}$, you will see that you get the correct sequence of dot products from the formula earlier).