Inner Products
Lay 6.1

1 The inner product

Definition 1.1. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$ with entries u_i and v_i respectively, we define their inner product

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n.$$

Note that by regarding $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices, this can be written as $\mathbf{u}^T\mathbf{v}$ (considered as a real number instead of a 1×1 matrix). Sometimes it is called the "dot product".

Example 1.2. If

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix},$$

then $\mathbf{u} \cdot \mathbf{v} = (1)(3) + (0)(2) + (4)(-4) = -13$.

Note that $\mathbf{u}$ and $\mathbf{v}$ have to have the same number of entries for the inner product to be defined. The dot product has some nice properties (these follow immediately from the definition):

Theorem 1.3.

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$;
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$;
- $\mathbf{u} \cdot \mathbf{u} \geq 0$, *with equality if and only if* $\mathbf{u} = \mathbf{0}$.

1.1 Length / Norms

The last property in the above theorem hints at an interpretation of $\mathbf{u} \cdot u$. We define $\|u\| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ to be the **norm** or length of $\mathbf{u}$. Then every vector except the zero vector has norm greater than 0. This also agrees via the pythagorean theorem with the usual notion of length in $\mathbb{R}^2$ or $\mathbb{R}^3$, when a vector (a, b, c) is identified with the point with coordinates $x = a, y = b, z = c$. For any scalar c and any vector $\mathbf{v}$, we have $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$. A vector with norm 1 is called a unit vector. By dividing any non-unit vector $\mathbf{v}$ by its length, we produce a unit vector in the "same direction as" $\mathbf{v}$ (in the sense that it lies on the line that goes through the point we identify with $\mathbf{v}$). I will show an example of this on the board. It is often convenient to turn a basis for a vector space into a basis of unit vectors. So this transformation is useful to us.

1.2 Distance

If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, we define their distance $\text{dist}(\mathbf{v}, \mathbf{w})$ to be the length of the vector $\mathbf{u} - \mathbf{v}$. Note this agrees with the usual notion in $\mathbb{R}^n$ for $n = 1, 2, 3$. I will describe this on the board in some detail (with pictures).

2 Orthogonality

A main feature of the inner product is that it lets us generalize the notion of "perpendicular" vectors from two and three dimensions.

Definition 2.1. We say $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

This agrees with the notion of perpendicularity in the spaces you are used to (see Figure 5 in Lay, and I will draw on the board as well).

Theorem 2.2. *Two vectors are orthogonal if and only if* $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Notice that this agrees with the usual notion of length in, say, $\mathbb{R}^2$. For instance, if $\mathbf{v} = (v_1, v_2)$, then $\mathbf{v} = v_1\mathbf{e}_1 + \mathbf{e}_2$ and we have

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 = \|v_1\mathbf{e}_1\|^2 + \|v_2\mathbf{e}_2\|^2.$$

Proof.

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}, \end{aligned}$$

and this last expression equals $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. □

3 Orthogonal Complements

Given a subspace W of $\mathbb{R}^n$, we want to define a notion of the "set of vectors orthogonal to W".

Definition 3.1. If $W \subseteq \mathbb{R}^n$ is a subspace, we say that $\mathbf{z} \in \mathbb{R}^n$ is orthogonal to W if $\mathbf{z}$ is orthogonal to every vector in W. The set of all vectors orthogonal to W is denoted by $W^\perp$.

Theorem 3.2. *Given a subspace W of $\mathbb{R}^n$, the following facts hold:*

- $\mathbf{x}$ *is in* $W^\perp$ *if and only if* $\mathbf{x}$ *is orthogonal to every vector in a set that spans* W*. This means we don't need to check the (infinitely many!) possible inner products of* $\mathbf{x}$ *with vectors of* W*. Instead we can check orthogonality with just these basis vectors.*
- $W^\perp$ *is a subspace of* $\mathbb{R}^n$*.*

Theorem 3.3. *Let* A *be an* $m \times n$ *matrix. Then*

$$(\mathrm{Row}A)^\perp = \mathrm{Nul}A, \quad \textit{and} \quad (\mathrm{Col}A)^\perp = \mathrm{Nul}A^T.$$

4 Angle

There is not a lot that we want to say about angles, except that the dot product between two vectors has a relationship to their angle (when treated as arrows / rays originating at the origin), at least for $\mathbb{R}^2$ and $\mathbb{R}^3$.

Proposition 4.1. *For vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, we have*

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos(\theta),$$

where θ *is the angle between the line segments from the origin to the two points identified with* $\mathbf{x}$ *and* $\mathbf{y}$*.*