Matrix Inversion
Reading: Lay 2.2

September 25, 2013

Before we start, please recall the definition of the identity matrix I_n.

1 Transposes

We are going to go over this very quickly (see the book for more information). If A is a matrix, then the **transpose** of A, written A^T, is the matrix whose rows are the columns of A. For instance, if

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 2 \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}.$$

2 The inverse

2.1 Informal introduction

Matrix multiplication is similar in many ways to multiplication of numbers– and different in many ways, too. Consider multiplication of real numbers for a moment. Say we have an equation like

$$5x = 10.$$

We want to "solve" for x. The way we do this is by multiplying each side by 1/5. The number 1/5 "undoes" multiplication by 5. That is, 1/5 is the number such that $5(1/5) = (1/5)5 = 1$. Using this on our equation gives

$$x = 10(1/5) = 2.$$

What about matrix equations? Say we are given an $n \times n$ matrix A and a vector $\mathbf{b}$, and want to solve the equation

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

for $\mathbf{x}$. It would be nice if there were some matrix A^{-1} with the property that $AA^{-1} = A^{-1}A = I_n$. Then we would be able to multiply both sides of (1) by A^{-1}:

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ \Longrightarrow I_n\mathbf{x} &= A^{-1}\mathbf{b} \\ \Longrightarrow \mathbf{x} &= A^{-1}\mathbf{b}. \end{aligned} \tag{2}$$

Note that in the above we multiply *on the left* by A^{-1}; remember that with matrices we have to keep track of the order of multiplication, since $AB \neq BA$ most of the time.

2.2 Invertibility

Given an $n \times n$ matrix A, if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, we call A^{-1} the **inverse** of A and say that A is **invertible**. Note that if A is invertible, its inverse A^{-1} is unique. The easiest way to see this is as follows: assume that there is another matrix B such that $AB = BA = I_n$. Then we have

$$\begin{aligned} AB &= I_n \\ \Longrightarrow A^{-1}AB &= A^{-1}I_n \\ \Longrightarrow (A^{-1}A)B &= A^{-1}I_n \\ \Longrightarrow I_nB &= A^{-1} \\ \Longrightarrow B &= A^{-1}. \end{aligned}$$

An invertible matrix is sometimes also called a **nonsingular** matrix– obviously this means that a noninvertible matrix is called a **singular matrix**.

2.3 A glimpse of things to come

Since Lay mentions this (see Lay 2.2, Theorem 4), I thought I should cover it here. If A is an $n \times n$ matrix, we can use the entries of A to calculate a number called the "determinant of A", denoted by $\det A$. We can use the determinant to tell right away whether A is invertible–it turns out that A is invertible if and only if $\det A \neq 0$.

The formula for the determinant is somewhat complicated when n is large; we'll talk more about this later in the semester. For a 2×2 matrix, though, the formula is very easy.

Definition 2.1. If A is a 2×2 matrix such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we define its determinant by the formula

$$\det A = ad - bc.$$

Theorem 2.2. *Let A be as in Definition 2.1. Then A is invertible if and only if* $\det A \neq 0$. *Moreover, a formula for the inverse is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We will talk more about determinants later in this course. The key point here is that you can easily tell whether a matrix is invertible just by looking at its determinant.

2.4 Invertibility and matrix equations

One thing to note is that if A is invertible, then for any vector $\mathbf{b}$ there is exactly one solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$. This solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

To see this, note that

$$A(A^{-1}\mathbf{b}) = (AA^{-1}\mathbf{b}) = I_n\mathbf{b} = \mathbf{b}.$$

So $A^{-1}\mathbf{b}$ solves the equation $A\mathbf{x} = \mathbf{b}$. On the other hand, say that we have some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. Multiplying on the left by A^{-1} gives $\mathbf{x} = \mathbf{b}$. So $A^{-1}\mathbf{b}$ is the only solution.

So consider a linear system whose coefficient matrix is A. If A is invertible, then there is exactly one solution to the equation $A\mathbf{x} = \mathbf{b}$, and we can find it using the inverse of A.

Example 2.3. Consider the linear system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 3. \end{aligned}$$

Solving this is the same as solving the matrix equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Using the formula from earlier in the notes, we see that $\det A = 1$. So A is invertible, and its inverse is

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore, a solution to the matrix equation is given by

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Thus our linear system has the unique solution $x_1 = -1$, $x_2 = 2$.

We should state some simple facts here:

Theorem 2.4.

- *If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
- *If A and B are $n \times n$ and invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
- *If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Note that in the second item above, we have to reverse the order of A and B when taking the inverse.

3 Elementary Matrices, Inversion Algorithm

An **elementary matrix** is a square matrix formed by performing a single elementary row operation on the identity matrix. There are three kinds of elementary matrices, corresponding to the three types of operations (add multiple of one row to another, swap rows, multiply a row by a constant). For instance, the following is one example of an elementary matrix:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Elementary matrices are important because performing an elementary row operation is the same as multiplying by an elementary matrix:

Theorem 3.1. *Let A be an $m \times n$ matrix. If an elementary row operation is performed on A to produce a new matrix B, then B can be written as EA, where E is the elementary matrix formed by performing the same elementary row operation on I_m.*

Example 3.2. Say that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

If we multiply row 2 of A by the number 3, we get the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

On the other hand, it is easy to see that $B = EA$, where E is as in equation (3) above (try this yourself!).

Remember that it is always possible to "undo" elementary row operations. That is, if we perform a row operation on A to produce a matrix B, we can perform another row operation on B to get A back. For instance, the way to undo the row operation in Example 3.2 is to multiply the second row of B by 1/3. This fact means that elementary matrices are invertible:

Theorem 3.3. *Every elementary matrix E is invertible. The inverse of E is the elementary matrix corresponding to the elementary row operation which turns E back into I_n.*

Example 3.4. The inverse to the matrix E from Example 3.2 is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The above theorem about inverting elementary matrices is actually pretty powerful, as surprising as that may be. It is an ingredient in the next theorem, which is not so hard to prove (see Lay for a proof). This theorem is extremely important, not just for what it says, but because we can turn it into an easy algorithm for finding inverse matrices. This will give you a way to calculate the inverse of A when A is bigger than 2×2!

Theorem 3.5. *An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n–that is, if and only if its RREF is I_n. In this case, any sequence of row operations which reduces A to I_n also transforms I_n into A^{-1}.*

3.1 The algorithm

The main idea behind our algorithm to invert matrices is to use the theorem above. Given a matrix A, we compute its RREF, while simultaneously performing the same row operations on the matrix I_n. If during our computation we manage to reduce A to I_n, then we stop and see what our row operations have done to I_n; they will have produced A^{-1}. If we get an RREF for A which is not I_n, then A is not invertible.

The easiest way to implement this is actually to change things slightly. We instead write the matrix

$$\begin{bmatrix} A & I_n \end{bmatrix},$$

that is, the matrix formed by putting the entries of A and I_n side-to-side inside one bigger matrix. We then row reduce *this* matrix, and see what happens. If A is row equivalent to I_n, then we will eventually produce the matrix

$$\begin{bmatrix} I_n & A^{-1} \end{bmatrix}.$$

Otherwise, if we do not get I_n in the left half of our bigger matrix when we compute the RREF, then A is not invertible.

Example 3.6. Given

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 0 & 0 \end{bmatrix},$$

What is A^{-1}? Let's first write

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

then compute the RREF.

The first pivot position of the matrix (4) is in the upper-left, and clearing out the column below it with two operations gives us

$$\begin{bmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -6 & -6 & -2 & 0 & 1 \end{bmatrix}$$

The next pivot position is the first 1 entry in the middle row. Clearing the column below it (with one operation) gives

$$\begin{bmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & -8 & 6 & 1 \end{bmatrix}. \tag{5}$$

We will clearly have three pivot positions in the left half of the matrix $\begin{bmatrix} A & I_3 \end{bmatrix}$, and so A must have three pivot positions; therefore, A is invertible. We wrap up the RREF computation by first clearing out the entries above the pivot in column 3 of (5):

$$\begin{bmatrix} 1 & 3 & 0 & -3 & 3 & 1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & -8 & 6 & 1 \end{bmatrix},$$

then by clearing out the entries above the pivot in column 2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & -8 & 6 & 1 \end{bmatrix},$$

and then finally multiplying the third row by $-1/6$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4/3 & -1 & -1/6 \end{bmatrix}.$$

Therefore, the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ -1 & 1 & 0 \\ 4/3 & -1 & -1/6 \end{bmatrix}.$$

Matrix inversion is another one of those things that will get easier with practice. You have already done a lot of RREF work in this course so far, which will help you with the technique.