# Vectors and Vector Equations
# Reading: Lay 1.3

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We've made some nice progress in the study of linear systems. To harness the full power of the theory we are developing, we will need to study objects called "vectors". So we are going to spend this lecture discussing vectors–what they are, how they behave, etc. This will be a nice break from the heavy computation that we dealt with last time.

## 1 Vectors in $\mathbb{R}^2$

A **column vector** is a matrix with only one column. Sometimes it helps to think of it as an ordered list which is arranged in a column. In this section we will talk about column vectors which have only two entries. Usually the entries of vectors are real numbers; we call the set of all real numbers by the name $\mathbb{R}$, and the set of all vectors with two real entries is called $\mathbb{R}^2$.

**Example 1.1.** The following are vectors in $\mathbb{R}^2$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ \pi \end{bmatrix}.$$

In general the vectors of $\mathbb{R}^2$ have the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for $v_1$ and $v_2$ in $\mathbb{R}$.

It may seem silly, but we will emphasize here again that vectors are defined as an *ordered* list. That is, two vectors are equal if and only if the corresponding entries are equal. So

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

but

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This is very important!!

Usually we represent a vector by a bold letter–for instance, $\mathbf{u}$.

## 1.1 Arithmetic in $\mathbb{R}^2$

We are going to define some basic arithmetic operations for vectors.

**Definition 1.2.** If $\mathbf{u}$ and $\mathbf{v}$ are elements of $\mathbb{R}^2$, we define $\mathbf{u} + \mathbf{v}$ to be the vector $\mathbf{w}$ whose entries are the sum of the corresponding entries in $\mathbf{u}$ and $\mathbf{v}$. That is,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

If $c$ is a constant, we define $c\mathbf{u}$ to be the vector whose entries are those of $\mathbf{u}$ multiplied by $c$:

$$c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c\, u_1 \\ c\, u_2 \end{bmatrix}.$$

We call a constant $c$ like the one in the above definition a **scalar**. This is just so we have a convenient word to distinguish vectors from the numbers which they take as their entries or which multiply them. We call the kind of multiplication that we just defined by the name "scalar multiplication." This means that we are multiplying vectors by scalars, not multiplying vectors by each other. We will not talk about what it means to multiply two vectors until much later in the course.

**Example 1.3.** Say we have two vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Then

$$3\mathbf{u} - 4\mathbf{v} = \begin{bmatrix} 3-8 \\ 12+12 \end{bmatrix} = \begin{bmatrix} -5 \\ 24 \end{bmatrix}.$$

### 1.2 Picturing $\mathbb{R}^2$

This section of the notes is going to be a little poor compared to the equivalent part of Lay because there are no pictures here. I will try to draw some pictures in class, and the figures in Lay are not so bad for when you are studying.

The nice thing about $\mathbb{R}^2$ is that it can be pictured as the plane. We identify a vector

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

as the point $(u_1, u_2)$; that is, the point in the plane with $x$-coordinate $u_1$ and $y$-coordinate $u_2$.

In this case, multiplication by scalars corresponds to translating the point along the line which goes through both $(0, 0)$ and $(u_1, u_2)$. There is also a nice picture for vector addition, called the **parallelogram rule for addition**. If you draw the parallelogram (as in Lay, Figure 3) which has three of its vertices $\mathbf{u}, \mathbf{v}$, and the origin, then $\mathbf{u} + \mathbf{v}$ is the fourth vertex.

There is another way to think about vector addition in $\mathbb{R}^2$ which does not really appear to be talked about in Lay, but which might help you to imagine what is happening. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

be vectors, and draw an arrow from the origin to the point $(u_1, u_2)$ and another arrow from the origin to the point $(v_1, v_2)$. Now take the arrow corresponding to $\mathbf{v}$ and move it so that the starting point of the arrow (the part without the "point") is at $(u_1, u_2)$. Now the place that this "moved" arrow is pointing will be the point in the plane which represents the vector $\mathbf{u} + \mathbf{v}$. Sometimes this is called the "tip to tail" or "head to tail" method.

## 2 Vectors in $\mathbb{R}^n$

A vector with $n$ real numbers as its entries is said to be an element of $\mathbb{R}^n$. The addition and scalar multiplication rules are defined just as in the case of $\mathbb{R}^2$. So, for instance,

$$c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c\,u_1 \\ c\,u_2 \\ \vdots \\ c\,u_n \end{bmatrix}.$$

Vectors in any $\mathbb{R}^n$ satisfy a number of algebraic properties, which I will list below in the same order as Lay does. It is easy to see why each of these properties is true. So it is probably more important to try to think for yourself why each of these properties is true, rather than trying to just memorize this whole list; then, when you are working with vectors, you will be able to use these identities more "naturally." Of course, you should try to study this material however works for you.

In what follows, we let $\mathbf{0}$ denote the vector in $\mathbb{R}^n$ whose entries are all zero.

## 2.1 Algebraic properties of vectors in $\mathbb{R}^n$

For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$, the following properties hold (note that $-\mathbf{u} = (-1)\mathbf{u}$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$;
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$;
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$;
8. $1\mathbf{u} = \mathbf{u}$.

The point of properties like the second one above are to allow us to write vector sums without parentheses–e.g., $\mathbf{u} + \mathbf{v} + \mathbf{w}$, and have it have unambiguous meaning. If property 2 did not hold, this would not be possible, because the sum would depend on the order in which we perform it.

## 2.2 Linear combinations

Say we have vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ (in $\mathbb{R}^n$) and constants $c_1, \ldots, c_p$. The vector

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

is called the **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

Note that the weights in a linear combination can be any numbers, including zero. Now, it turns out that linear combinations are related to the problems we talked about in the last couple of lectures. What do I mean? Say we have three vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

and I ask you a question: can we write $\mathbf{b}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$? This question is the same as asking whether there are weights $x_1, x_2$ such that

$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{b}$$

or, to write out the equations in full,

$$\begin{aligned} u_1x_1 + v_1x_2 &= b_1 \\ u_2x_1 + v_2x_2 &= b_2. \end{aligned}$$

So what we are really asking is whether a certain linear system is consistent.

**Example 2.1.** Say we have the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Can we write $\mathbf{b}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$? Well, we can if and only if the following linear system is consistent:

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 - 2x_2 &= 0. \end{aligned}$$

We know how to answer this question using the RREF techniques from last time. I won't work out the RREF here (you should try on your own), but the answer is yes, we can write $\mathbf{b}$ as such a linear combination.

Let's introduce some notation before we summarize what we've learned. If $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ are vectors, let

$$\begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_q \end{bmatrix}$$

denote the matrix whose columns are $\mathbf{w}_1, \ldots, \mathbf{w}_q$ respectively.

**Theorem 2.2.** *A vector equation*

$$x_1\mathbf{v}_1 + \ldots x_p\mathbf{v}_p = \mathbf{b}$$

*has the same solution set (that is, is solved by the same values of $x_1, \ldots, x_p$) as the linear system whose augmented matrix is*

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p & \mathbf{b} \end{bmatrix}.$$

*In particular, we can write $\mathbf{b}$ as a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ if and only if this linear system is consistent.*

# 3 Span

One question that you might now ask is: given a set of vectors $\{\mathbf{v}_i\}$ (each $\mathbf{v}_i$ is an element of $\mathbb{R}^n$ for the same number $n$), what does the set of all possible linear combinations of vectors in this set look like?

**Definition 3.1.** Given vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ in $\mathbb{R}^n$, we define **span**$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ to be the set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, the span is the set of all vectors $\mathbf{y}$ which can be written

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p.$$

By Theorem 2.2, we can tell whether a vector is in the span by determining whether an appropriate linear system is consistent or inconsistent. There are some ways to interpret the span geometrically, which Lay discusses in more detail. For instance, the span of a single vector in $\mathbb{R}^2$ can be represented as a line in the plane which passes through the origin.