Determinants, Part One

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Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Remember that before we defined a number called the determinant of A:

$$\det A = ad - bc,$$

with a special property: whenever $\det A \neq 0$, we were guaranteed A was invertible, and in fact had the explicit formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a. \end{bmatrix} \tag{1}$$

If you don't believe (1), it is easy to convince yourself; just compute the matrix products AA^{-1} and $A^{-1}A$ using the above formula for A^{-1}, and you will see that both products give the identity matrix I_2.

Today we will define the determinant for $n \times n$ matrices. Our goal is to define det so that the following property holds for all square matrices A:

- A is invertible if and only if $\det A \neq 0$.

This lecture is going to be devoted to just engineering a definition of determinant, based on the one we have for 2×2 matrices, such that it is at least plausible that the above property holds. Next time, we will explore more closely the relationship to invertibility.

1 Definition of Determinant: 3×3

Let's start by looking at an invertible 3×3 matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We need to remember two important properties of invertible matrices in what follows:

1. If an $n \times n$ matrix B is invertible, it has n pivot positions (in particular, it has no columns or rows which are all zero);

2. If B is invertible, any matrix we get by performing row operations on B is still invertible (since we don't change the number of pivots).

Since the first column of A is not the zero column, it has a nonzero entry; we might as well assume that $a_{11} \neq 0$ (we can always do a row interchange to make this true). Let's perform some row operations on A: first multiply rows 2 and 3 by a_{11} to get the matrix

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix},$$

then add $-a_{21}$ times row 1 to row 2, and $-a_{31}$ times row 1 to row 3, to get

$$A'' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}.$$

Now, A'' is still invertible by our "two properties", so it has a pivot position in the second column; since a_{11} is already a pivot, this pivot position has to lie in either the second or third rows. This means that either the $(2,2)$ entry or the $(3,2)$ entry of A'' is nonzero; we might as well assume that $A''_{22} \neq 0$ (otherwise we could just interchange rows).

So we do to A'' something similar to what we did in the beginning to A. We multiply row 3 of A'' by the (nonzero!) number $a_{11}a_{22} - a_{12}a_{21}$, then we add the appropriate multiple of row 2 to row 3 to clear out the second entry of row 3 (the "appropriate multiple") here is $a_{11}a_{32} - a_{12}a_{31}$. Rather than work this calculation, we just show the result:

$$A''' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where

$$\Delta = (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}).$$

Note that A''' is invertible (because A is), so $\Delta \neq 0$.

We will define $\det A$ to be the number Δ above. This gives us a definition of det for 3×3 matrices, and we see that $\det A \neq 0$ when A is invertible. We will soon see that the converse holds; that is, if $\det A \neq 0$, then A is invertible. First, we will cast Δ in a more illuminating form which will help us guess a generalization to bigger matrices. Consider the 2×2 submatrix of A obtained by deleting the first row and column of A; we denote this submatrix by A_{11}:

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

Then $\det A_{11} = a_{22}a_{33} - a_{32}a_{23}$. So the first term of Δ above is actually equal to $a_{11} \det A_{11}$.

Similarly, if we let A_{12} denote the submatrix of A obtained by deleting the first row and second column of A, and we let A_{13} be defined similarly, we have

$$\det A = \Delta = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}. \tag{2}$$

2 Determinants for general matrices

Our philosophy for defining determinants will be inspired by the form (2). We will give a "recursive" definition: the determinants of 3×3 matrices are defined using determinants for 2×2 matrices as in (2); the determinants of 4×4 matrices are defined using determinants of 3×3 matrices; etc.

This definition will actually also encompass our previous definition for det of a 2×2 matrix, as long as we set $\det \begin{bmatrix} a \end{bmatrix} = a$ for a 1×1 matrix.

Definition 2.1. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ matrix (this notation just means that we will denote the entry which lies in the ith row and jth column of A

by a_{ij}). If $n = 1$, we define $\det A = a_{11}$, the only entry of A. Otherwise, we set

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \ldots + (-1)^{1+n}a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}.\end{aligned}$$

Example 2.2. Let

$$A = \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 0 & 0 & 4\end{bmatrix}.$$

Then we have

$$\begin{aligned}\det A &= 1\cdot\det\begin{bmatrix}4 & 6\\ 0 & 4\end{bmatrix} - 2\cdot\det\begin{bmatrix}2 & 6\\ 0 & 4\end{bmatrix} + 3\cdot\det\begin{bmatrix}2 & 4\\ 0 & 0\end{bmatrix} \\ &= 1\cdot 16 - 2\cdot 8 + 3\cdot 0 = 0.\end{aligned}$$

Example 2.3. Let

$$A = \begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 0\\ 0 & 3 & 1\end{bmatrix}.$$

Then

$$\begin{aligned}\det A &= 1\cdot\det\begin{bmatrix}3 & 0\\ 3 & 1\end{bmatrix} - 1\cdot\det\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + 1\cdot\det\begin{bmatrix}0 & 3\\ 0 & 3\end{bmatrix} \\ &= 1\cdot 16 - 2\cdot 8 + 3\cdot 0 = 0.\end{aligned}$$

3 Cofactor expansion

Computing determinants of matrices bigger than 3×3 using the definition rapidly gets horrible. A number of theorems and techniques exist to make computing determinants of bigger matrices more palatable. The first we will see is "cofactor expansion".

Definition 3.1. Let A be an $n\times n$ matrix. Analogous to before, define A_{ij} to be the submatrix of A produced by deleting the ith row and jth column of A. Then the (i, j) cofactor of A, denoted by C_{ij}, is defined by

$$C_{ij} = (-1)^{i+j}\det A_{ij}.$$

Definition 3.1 gives us another way to rewrite the definition of a determinant:

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

This notation is suggestive, and might lead you to believe that the determinant could be computed using the cofactors C_{ij} instead, where $i \neq 1$. In fact, this is true and more. We could have performed what is called a **cofactor expansion** using any row *or column* of the matrix, as we see in the following theorem:

Theorem 3.2. *Let A be an $n \times n$ matrix. Then $\det A$ can be computed using cofactor expansion along any row of the matrix. The cofactor expansion along the ith row is given by*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots a_{in}C_{in}.$$

On the other hand, we could also compute using cofactor expansion along any column of the matrix. The cofactor expansion along the jth column is given by

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Justifying this theorem is somewhat complicated, so we will just take it for granted without proof.

The cofactor expansion is quite useful for computing determinants in the case that a given row or column has a lot of zero entries.

Example 3.3. Compute $\det A$ if

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Performing cofactor expansion along the bottom row, we see

$$\begin{aligned}
\det A &= 0 \cdot \det \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 2 & 2 & 2 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 1 & 3 & 1 \\ 4 & 6 & 1 \\ 2 & 2 & 2 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix} \\
&\quad + 1 \cdot \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 2 & 2 \end{bmatrix} \\
&= 0 + 0 + 0 + 1 \cdot \left(1 \det \begin{bmatrix} 5 & 6 \\ 2 & 2 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix} + 3 \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix} \right) \\
&= -2 - 2(-4) + 3(-2) = 0.
\end{aligned}$$

We finish up today by showing a nice way to compute the determinants of upper or lower triangular matrices.

Example 3.4. Using cofactor expansion on the first column of the following 3×3 matrix,

$$\begin{aligned}
\det \begin{bmatrix} 3 & 3 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} &= 3 \cdot \det \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \\
&= 3 \cdot 2 \cdot \det \begin{bmatrix} 4 \end{bmatrix} = 3 \cdot 2 \cdot 4 = 24.
\end{aligned}$$

The same technique works on any upper or lower triangular matrix.

Theorem 3.5. *If A is upper or lower triangular, then* $\det A$ *is equal to the product of the entries on the main diagonal of A.*

This theorem will be the basis of a simpler method of computing determinants than the ones we have seen so far; we will discuss this in a future lecture.