Practice Exam Solutions

Note: most of the problems have many equally valid solutions!

Problem 1

(a) True. This is the rank theorem.

(b) False. We have spent a lot of time talking about how many matrices are not diagonalizable. One example is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(c) False. A regular stochastic matrix P is allowed to have 0 in some entries, as long as P^k does not have 0 entries for some k.

(d) False. An orthogonal matrix U satisfies $U^T = U^{-1}$. This does not imply that $U^T = U$. One example appears in the "orthogonal diagonalization" problem at the end of these problems.

(e) True. $B^{-1}A^{-1}AB = B^{-1}B = I$.

Problem 2

Write the augmented matrix of the system:

$$\begin{bmatrix} 1 & h & 1 \\ 3 & 12 & 1 \end{bmatrix}.$$

One step of row reduction gives

$$\begin{bmatrix} 1 & h & 1 \\ 0 & 12 - 3h & -2 \end{bmatrix}.$$

Clearly this will be the matrix of a consistent system if and only if $h \neq 4$. So the system has no solution for the value $h = 4$.

Problem 3

(a) What the problem is really asking is for the matrix of T with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$. Now,

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$$

is the right choice.

Problem 3

(b) With some practice, you can simply read off the matrix entries from the polynomial. But if you are uncomfortable with this, let's take the long way. Set

$$B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Then $\mathbf{x}^T B \mathbf{x} = a x_1^2 + 2b x_1 x_2 + c x_2^2$. Matching coefficients with Q in the problem gives $a = 2$, $b = 3$, $c = 2$. So

$$B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

Problem 4

This is just a matter of solving the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. We calculate

$$A^T A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 16 \\ 64 \end{bmatrix}.$$

Row reduction on augmented matrix $\begin{bmatrix} A^T A & A^T\mathbf{b} \end{bmatrix}$ gives

$$\begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 2 \end{bmatrix},$$

or

$$\hat{\mathbf{x}} = \begin{bmatrix} -2/3 \\ 2 \end{bmatrix}.$$

Problem 5

Note that

$$det(A) = 1, \quad \det(B) = ad - bc, \quad \det(A+B) = (1+a)(1+d) - bc.$$

In particular, $\det(A+B) = ad + a + d + 1 - bc$, and $\det(A) + \det(B) = 1 + ad - bc$. These two expressions are equal if and only if $a + d = 0$.

Problem 6

First let's find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then normalize. Start with $\mathbf{u}_1 = \mathbf{v}_1$.

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Similarly, $$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} -1/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Now,

$$\|\mathbf{u}_1\| = 2; \quad \|\mathbf{u}_2\| = \sqrt{2}; \quad \|\mathbf{u}_3\| = 1/2.$$

So an orthonormal basis is

$$\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}.$$

Problem 7

$$\mathbf{x}_1 = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

$$\mathbf{x}_2 = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5/12 \\ 7/12 \end{bmatrix}.$$

Problem 7

To find $\mathbf{q}$, write $P\mathbf{q} = \mathbf{q} \Longrightarrow (P - I)\mathbf{q} = \mathbf{0}$. We solve the system whose augmented matrix is

$$\begin{bmatrix} -1/2 & 1/3 & 0 \\ 1/2 & -1/3 & 0 \end{bmatrix}.$$

Clearly this is solved by

$$c \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Normalizing to get the sum of entries 1 gives

$$\mathbf{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}.$$

Problem 8

Let's use the equation $AA^{-1} = I$ with our formulas for A and A^{-1}:

$$\begin{aligned} I_n &= (I_n + \mathbf{v}\mathbf{v}^T)(I_n + \alpha\mathbf{v}\mathbf{v}^T) \\ &= I_n + \mathbf{v}\mathbf{v}^T + \alpha\mathbf{v}\mathbf{v}^T + \alpha\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T \\ &= I_n + (1 + \alpha + \alpha\|\mathbf{v}\|^2)\mathbf{v}\mathbf{v}^T \\ \Longrightarrow 0_n &= \mathbf{v}\mathbf{v}^T \left[1 + \alpha\left(1 + \|v\|^2\right)\right], \end{aligned}$$

where 0_n is the $n \times n$ zero matrix. Now, $\mathbf{v}\mathbf{v}^T$ is not the zero matrix, so this implies that the sum of numbers is square brackets is zero, or equivalently

$$\alpha = \frac{-1}{1 + \|v\|^2}.$$

It would have been ok to leave $\|\mathbf{v}\|^2$ as $\mathbf{v}^T\mathbf{v}$.

Problem 9

To find a basis for $\operatorname{col} A$, we look for pivot columns. Adding -2 times row 1 to row 3 gives

$$\begin{bmatrix} 1 & 3 & -2 & 1 & -5 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 2 & 2 \end{bmatrix}$$

taking the row reduction one step further gives

$$\begin{bmatrix} 1 & 3 & -2 & 1 & -5 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see the first and third columns of A are pivot columns.

So a basis for $\operatorname{col} A$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix} \right\}.$$

A basis for $\operatorname{row} A$ is provided by

$$(1, 3, -2, 1, -5), \quad (0, 0, -1, 2, 2).$$

To find a basis for $\operatorname{nul} A$, we have to finish row reduction. Actually, we want to row reduce matrix augmented by $\mathbf{0}$, but the zero column remains unchanged under row operations, so we can use our work so far.

Problem 9

Finishing row reduction of augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -3 & -9 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the null space is all vectors of form

$$\left\{ \begin{bmatrix} -3x_2 + 3x_4 + 9x_5 \\ x_2 \\ 2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} : \quad x_2, x_4, x_5 \in \mathbb{R} \right\}$$

Problem 9

So a basis for $\operatorname{nul} A$ is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the dimensions of the column space and row space are the same, as expected!

Problem 10

The characteristic polynomial is

$$(2-\lambda)(2-\lambda)-9=\lambda^2-4\lambda-5=(\lambda-5)(\lambda+1).$$

So the eigenvalues are $\lambda_1=-1$ and $\lambda_2=5$. Eigenvectors:

$$A+I=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}\implies \mathbf{v}_1=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$$

and

$$A-5I=\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}\implies \mathbf{v}_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}.$$

Problem 10

So $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Note that since P is orthogonal, $P^{-1} = P^T$.