Enrique Areyan - M464 - Exam 1 - Spring 2014 (8) (1)

(1) (a) State 2 is absorbing since $P_{2,2}^{(n)} = 1$, for every $n \in \{1,2,3,\dots\}$ 5
and $P_{2,i}^{(n)} = 0$, for all other states.

(b) Let $T = \min\{n \geq 1 : X_n = 2\}$ and $X_0 = 0$. Wish to find:

$Pr\{X_3 = 0 \mid T > 3\}$ = probability of being in state 0 at time 3 given (and starting at 0). the chain is not in the absorbing state. By definition, this is just the $P_{0,0}^{(3)} = \frac{284}{100}$, i.e., the 3-step transition starting from 0 and ending back at 0. If the chain is not in absorbing state then we just compute the 3-step transition in this case (it is given!) 1

(c) Let $T = \min\{n \geq 1 ; X_n = 2\}$. We wish to find

$$Pr\{X_{T-1} = 0 \mid X_0 = 0\} = \frac{Pr\{X_{T-1} = 0, X_0 = 0\}}{Pr\{X_0 = 0\}} = Pr\{X_{T-1} = 0, X_0 = 0\},$$

Since we know $X_0 = 0$ so $Pr\{X_0 = 0\} = 1$. Now, we can compute:

always true by defn.

$$Pr\{X_T = 2, X_{T-1} = 0, X_0 = 0\} = Pr\{X_T = 2 \mid X_{T-1} = 0, X_0 = 0\} Pr\{X_{T-1} = 0, X_0 = 0\}$$

$$= Pr\{X_T = 2 \mid X_{T-1} = 0\} \cdot Pr\{X_{T-1} = 0 \mid X_0 = 0\} Pr\{X_0 = 0\}$$ Markov Property

$$= Pr\{X_T = 2 \mid X_{T-1} = 0\} \cdot Pr\{X_{T-1} = 0 \mid X_0 = 0\}$$ Since $Pr\{X_0 = 0\} = 1$

$$= 0.6 \cdot Pr\{X_{T-1} = 0 \mid X_0 = 0\}$$ Prob. transitions

Let $u_i = Pr\{X_{T-1} = 0 \mid X_0 = i\}$. Then, By first step analysis:

$$\begin{cases} u_0 = 0.6u_0 + 0.2u_1 + 0.2u_2 \\ u_1 = 0.2u_0 + 0.5u_1 + 0.3u_2 \\ u_2 = 1 \text{ (already absorb)} \end{cases} \qquad \begin{cases} 0.4u_0 = 0.2u_1 + 0.2 \\ 0.5u_1 = 0.2u_0 + 0.3 \end{cases}$$ 2

These have solution $u_0 = u_1 = 1$.

$$\begin{cases} \frac{4}{10}u_0 = \frac{2}{10}u_1 + \frac{2}{10} \\ \frac{5}{10}u_1 = \frac{2}{10}u_0 + \frac{3}{10} \end{cases} \Rightarrow \begin{cases} 4u_0 = 2u_1 + 2 \Rightarrow u_0 = \frac{1}{2}u_1 + \frac{1}{2} \\ 5u_1 = 2u_0 + 3 \Rightarrow u_1 = \frac{2}{5}u_0 + \frac{3}{5} \end{cases}$$

$$u_1 = \frac{2}{5}\left(\frac{1}{2}u_1 + \frac{1}{2}\right) + \frac{3}{5} = \frac{1}{5}u_1 + \frac{1}{2} + \frac{3}{5} \Rightarrow \frac{4}{5}u_1 = \frac{5+6}{10} \Rightarrow u_1 = \frac{11}{10}\cdot\frac{4}{5}$$

$\Rightarrow u_1 = \frac{44}{50} \Rightarrow$ ran out of time, just solve for u_0

2) We can model this situation as a M.C. with Transition matrix:

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1/2	0	1/2	0	0	0
2	0	1/2	0	1/2	0	0
3	0	0	1/2	0	1/2	0
4	0	0	0	1/2	0	1/2
5	0	0	0	0	0	1

For the purposes of our analysis we can consider states 0 and 5 to be absorbing so we can compute the probability we want:

Let $T = \min\{n \geq 1 ; X_T = 0\}$, i.e., first time the rat finds food.

Now, let $u_i = Pr\{X_T = 0 | X_0 = i\}$. Then, By first-step analysis:

$$\begin{cases} u_0 = 1 \\ u_1 = \frac{1}{2}u_0 + \frac{1}{2}u_2 \\ u_2 = \frac{1}{2}u_1 + \frac{1}{2}u_3 \\ u_3 = \frac{1}{2}u_2 + \frac{1}{2}u_4 \\ u_4 = \frac{1}{2}u_3 + \frac{1}{2}u_5 \\ u_5 = 0 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{1}{2} + \frac{1}{2}u_2 \\ \frac{3}{2}u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_3 \Rightarrow u_3 = \frac{1}{2}u_1 \\ u_3 = \frac{1}{2}u_2 + \frac{1}{2}\left(\frac{1}{2}u_3\right) = \frac{1}{2}u_2 + \frac{1}{4}u_3 \Rightarrow \frac{3}{4}u_3 = \frac{1}{2}u_2 \\ u_4 = \frac{1}{2}u_3 \end{cases}$$

$$\Downarrow$$

$$\frac{3}{2}u_3 = u_2$$

So we have:

$$\begin{cases} u_1 = \frac{1}{2} + \frac{1}{2}u_2 \\ u_2 = \frac{3}{2}u_3 \\ u_3 = \frac{1}{2}u_1 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{4}u_1\right) \Rightarrow u_1 = \frac{1}{2} + \frac{3}{8}u_1 \Rightarrow \frac{5}{8}u_1 = \frac{1}{2} \quad (*) \\ u_2 = \frac{3}{2}\left(\frac{1}{2}u_1\right) \Rightarrow u_2 = \frac{3}{4}u_1 \end{cases}$$

$(*) \Rightarrow \boxed{u_1 = \frac{4}{5}}$ the probability of first reaching the food is the probability of reaching absorbing state 0 when starting from 3, i.e.,

$$Pr\{X_T = 0 | X_0 = 3\} = u_3 = \frac{1}{2}u_1 = \frac{1}{2}\frac{4}{5} \Rightarrow \boxed{u_3 = \frac{2}{5}}$$

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3) (a) We can model this situation as a M.C. $\langle X_n, n \geq 0 \rangle$, where X_n = # of balls in urn A at time n. The transition matrix is given by: (2)

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

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The prob. correspond to selecting a random urn and, if the number of balls in urn A are between 1 and 3, we can either add or remove a ball. If the number is 0, then we can either stay at zero if urn A is selected again or go to one. Finally, if in state 5, then either we select the other urn and stay in 5 or go down to four. Let us find the limiting distribution $\pi P = \pi$ and $\sum_{i=0}^{5} \pi_i = 1$

$$\begin{cases} \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 = \pi_0 \\ \frac{1}{2}\pi_0 + \frac{1}{2}\pi_2 = \pi_1 \\ \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 = \pi_2 \\ \frac{1}{2}\pi_2 + \frac{1}{2}\pi_4 = \pi_3 \\ \frac{1}{2}\pi_3 + \frac{1}{2}\pi_5 = \pi_4 \\ \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5 = \pi_5 \end{cases} \qquad \begin{cases} \pi_0 = \pi_1 \\ \pi_1 = \pi_2 \\ \pi_2 = \pi_3 \\ \pi_3 = \pi_4 \\ \\ \pi_4 = \pi_5 \end{cases}$$

$\Rightarrow \pi_0 = \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5$

And so, $\sum_{i=0}^{5} \pi_i = 1 \Rightarrow$

$6\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{6}$ ✓

So, the long run fraction of time urn A is empty is $\boxed{\frac{1}{6}}$

Why is this $= \pi_0$?

(b) The chain is regular. Checking our sufficient condition:

i) $P_{00} = \frac{1}{2} > 0 \Rightarrow \exists$ state i : $P_{ii} > 0$ (take $i = 0$ or 5)

ii) $\exists$ a path between any two states; just take $i \to i+1 \to \cdots \to j$ (for $i < j$) ✓ or the reverse path in case $i > j$.

4) First note that this is a regular matrix because:

i) $P_{A,A} = 0.6 > 0$, so $\exists i : P_{ii} > 0$ (in fact any i works).

ii) $\exists$ a path from any state i to any other state j, just take $P_{i,j} > 0$ (all entries positive).

By theorem 1.1, we know there exists a unique limiting distribution satisfying $\pi P = \pi$ and $\pi_A + \pi_B + \pi_C = 1$. Moreover, we know that this limiting distribution represents the fraction of time the chain is in a particular state. Therefore, the answer would be π_A in the following equations:

$$\begin{cases} \pi_A 0.6 + \pi_B 0.1 + \pi_C 0.1 = \pi_A \\ \pi_A 0.2 + \pi_B 0.7 + \pi_C 0.1 = \pi_B \\ \pi_A 0.2 + \pi_B 0.2 + \pi_C 0.8 = \pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases}$$

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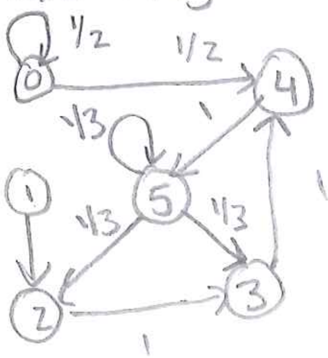
5) A state diagram for this chain is:



Communicating classes:

$0 \to 4$ but 4 does not com. with 0. So 0 and 4 are in different classes. In fact, no other state comm. with 0 so $\{0\}$ is one com. class.

$4 \to 5$ and $5 \to 3 \to 4$, so $4 \leftrightarrow 5$. Also, $5 \to 2$, $2 \to 3 \to 5$, so $2 \to 5$. Hence, $5 \leftrightarrow 2$. By transitivity of com. relation: $4 \leftrightarrow 5$ and $5 \leftrightarrow 2 \Rightarrow 4 \leftrightarrow 2$. Finally $4 \to 5 \to 3$ and $3 \to 4$, so $4 \leftrightarrow 3$. Therefore, another com. class is $\{2,3,4,5\}$. The last com. class is $\{1\}$, which does not communicate with any state.

Periods: By definition $d(i) := \gcd\{n \geq 1;\ P_{ii}^{(n)} > 0\}$. We also know that period is a class property, so it suffices to check period of one member of each class to know all periods. We begin →

For class $\{0\}$

$d(0) = \gcd\{1,2,3,\dots\} = 1$; since $\gcd(2,3) = 1$

So class $\{0\}$ is aperiodic.

For class $\{2,3,4,5\}$

$d(5) = \gcd\{1,2,3,\dots\} = 1$; since $\gcd(2,3) = 1$

So class $\{5\}$ is also aperiodic

For class $\{1\}$,

$d(1) = \gcd\{0\}$ thus, by definition the period is

also ~~one~~. In fact this chain is aperiodic.

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