M464 - Introduction To Probability II - Homework 10

Enrique Areyan
April 3, 2014

Chapter 5

Exercises

5.3 Defects (air bubbles, contaminants, chips) occur over the surface of a varnished tabletop according to a Poisson process at a mean rate of one defect per top. If two inspectors each check separate halves of a given table, what is the probability that both inspectors find defects?

Solution: Let $\{N(A); A \in \mathcal{A}\}$ be a Poisson point process of intensity 1 describing the distribution of defects on the tabletop. Then, let $\{A_1, A_2\}$ be a partition for the entire table A, i.e., $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, each corresponding to half of the table. Hence, $|A_1| = |A_2| = \dfrac{1}{2}$. By postulates (2) and (3),

(a) $N(A_1) \sim Pois(1\frac{\text{defects}}{\text{top}}|A_1| \text{ top}) = Pois(\frac{1}{2} \text{ defects})$

(b) $N(A_2) \sim Pois(1\frac{\text{defects}}{\text{top}}|A_1| \text{ top}) = Pois(\frac{1}{2} \text{ defects})$

(c) $N(A_1)$ is independent of $N(A_2)$

Therefore, noting that $N(A_i) \geq 1$ for $i = 1, 2$ is the event that there is at least one defect in half i, we can compute the probability that both inspectors find defects $Pr\{N(A_1) \geq 1, N(A_2) \geq 1\}$ as follows:

$$
\begin{aligned}
Pr\{N(A_1) \geq 1, N(A_2) \geq 1\} &= Pr\{N(A_1) \geq 1\}Pr\{N(A_2) \geq 1\} && \text{by independence of } N(A_1) \text{ with } N(A_2) \\
&= [1 - Pr\{N(A_1) = 0\}][1 - Pr\{N(A_2) = 0\}] && \text{complementary events} \\
&= [1 - e^{-\frac{1}{2}}\tfrac{(\frac{1}{2})^0}{0!}\}][1 - e^{-\frac{1}{2}}\tfrac{(\frac{1}{2})^0}{0!}\}] && \text{poisson p.m.f} \\
&= [1 - e^{-\frac{1}{2}}]^2 && \text{simplyfing} \\
&= \boxed{0.154818}
\end{aligned}
$$

6.1 Customers demanding service at a central processing facility arrive according to a Poisson process of intensity $\theta = 8$ per unit time. Independently, each customer is classified as *high priority* with probability $\alpha = 0.2$, or *low probability* with probability $1 - \alpha = 0.8$. What is the probability that three high priority and five low priority customers arrive during the first unit of time?

Solution: By results worked in the section *Thinned Poisson Processes*, we know that arrivals from each type of customer form a Poisson process and that the processes are independent of each other. Hence, let:

(a) $X_h(t)$ = number of arrivals from *high priority* customers up to time t. During the first unit of time $t = 1$ and thus:

$$X_h(1) \sim Pois(\frac{2}{10} \cdot 8\frac{\text{cust}}{\text{time}} \cdot 1 \text{ time}) = Pois(1.6 \text{ cust})$$

(b) $X_l(t)$ = number of arrivals from *low priority* customers up to time t. During the first unit of time $t = 1$ and thus:

$$X_l(1) \sim Pois(\frac{8}{10} \cdot 8\frac{\text{cust}}{\text{time}} \cdot 1 \text{ time}) = Pois(6.4 \text{ cust})$$

The probability we wish to find is $Pr\{X_h(1) = 3, X_l(1) = 5\}$. We computed as follow:

$$
\begin{aligned}
Pr\{X_h(1) = 3, X_l(1) = 5\} &= Pr\{X_h(1) = 3\}Pr\{X_l(1) = 5\} && \text{by independence of } X_h \text{ with } X_l \\
&= \left[e^{-1.6}\tfrac{(1.6)^3}{3!}\right]\left[e^{-6.4}\tfrac{(6.4)^5}{5!}\right] && \text{poisson p.m.f} \\
&= \tfrac{e^{-8}(1.6)^3(6.4)^5}{5!3!} && \text{simplifying} \\
&= \boxed{0.02049139}
\end{aligned}
$$

6.4 Men and women enter a supermarket according to independent Poisson processes having respective rates of two and four per minute.

(a) Starting at an arbitrary time, what is the probability that at least two men arrive before the first women arrives?
Solution: Let $X(t)$ = number of men entering the supermarket up to minute t. Then, $X(t) \sim Pois(2t)$ because men enter the supermarket according to a Poisson process. Let W_1 be the waiting time for the first women to arrive to the supermarket. Since we know that women enter according to a Poisson process, by theorem 3.1: $W_1 \sim Exp(4t)$. Now, we wish to compute the probability: $Pr\{X(W_1) \geq 2\}$, i.e., the probability that 2 or more men enter the supermarket before the first women arrives. But then:

$$\begin{aligned}
Pr\{X(W_1) \geq 2\} &= \int_0^t Pr\{X(t) \geq 2 | W_1 = t\} Pr\{W_1 = t\} dt && \text{law of total probability} \\
&= \int_0^t [1 - Pr\{X(t) < 2\}] 4e^{-4t} dt && \text{complementary event and exponential pdf} \\
&= \int_0^t [1 - Pr\{X(t) = 0\} - Pr\{X(t) = 1\}] 4e^{-4t} dt && \text{equivalent event} \\
&= \int_0^t [1 - e^{-2t}\tfrac{(2t)^0}{0!} - e^{-2t}\tfrac{(2t)^1}{1!}] 4e^{-4t} dt && \text{poisson pmf} \\
&= 4\int_0^t e^{-4t} - e^{-6t} - 2te^{-6t} dt && \text{algebra} \\
&= \frac{1}{9}\left[e^{-6t}(8 - 9e^{2t} + 12t)\right]_0^\infty && \text{solving integral (by parts)} \\
&= \frac{1}{9}(0 - (8 - 9)) && \text{plugging limit values} \\
&= \boxed{\frac{1}{9}}
\end{aligned}$$

(b) What is the probability that at least two men arrive before the third woman arrives?
Solution: By the same reasoning and letting W_3 be the arrival time of the third women, we wish to compute $Pr\{X(W_3) \geq 2\}$. By theorem 3.1, we know the distribution of W_3, $f_{W_3}(t) = \dfrac{4^3}{(3-1)!} t^{3-1} e^{-4t} = 32t^2 e^{-4t}$.
Again, by law of total probability:

$$\begin{aligned}
Pr\{X(W_3) \geq 2\} &= \int_0^t Pr\{X(t) \geq 2 | W_3 = t\} Pr\{W_3 = t\} dt && \text{law of total probability} \\
&= \int_0^t [1 - Pr\{X(t) < 2\}] 32t^2 e^{-4t} dt && \text{complementary event and exponential} \\
&= \int_0^t [1 - Pr\{X(t) = 0\} - Pr\{X(t) = 1\}] 32t^2 e^{-4t} dt && \text{equivalent event} \\
&= \int_0^t [1 - e^{-2t}\tfrac{(2t)^0}{0!} - e^{-2t}\tfrac{(2t)^1}{1!}] 32t^2 e^{-4t} dt && \text{poisson pmf} \\
&= 32\int_0^t t^2 e^{-4t} - t^2 e^{-6t} - 2t^3 e^{-6t} dt && \text{algebra} \\
&= \frac{1}{27}\left[e^{-6t}(-27e^{2t}(1 + 4t + 8t^2) + 16(1 + 6t + 18t^2 + 18t^3))\right]_0^\infty && \text{solving integral (by parts)} \\
&= \frac{1}{27}(0 - (-27(1) + 16(1)) && \text{plugging limit values} \\
&= \boxed{\frac{11}{27}}
\end{aligned}$$

Problems

5.6 Suppose that stars are distributed in space following a Poisson point process of intensity λ. Fix a star *alpha* and let R be the distance from alpha to its nearest neighbor. Show that R has the probability density function

$$f_R(x) = (4\lambda\pi x^2) e^{\frac{-4\lambda\pi x^3}{3}}, \qquad x > 0$$

Solution: Let $R =$ the distance from alpha to its nearest neighbor. If we can find the cumulative distribution function of R, then by taking its derivative we find its probability distribution function. Finding the c.d.f is relatively easy noting that $Pr\{R > x\} = Pr\{N(A) = 0\}$, where $N(A)$ corresponds to the poisson point process of the distribution of starts on region A of space where A is the sphere of radius x centered at alpha. In other words, the probability that the nearest start to alpha is as far as x units of distance is the same as having seen no starts in the sphere A of radius x centered at alpha. By postulate (i) (page 312), we know that $N(A) \sim Pois(\lambda|A|)$. In this case the region of space we are considering is an sphere of radius x (arbitrarily fixing alpha to be at the origin) and thus, $|A| = vol(A) = \frac{4}{3}\pi x^3$. Hence,

$$N(A) \sim Pois(\lambda|A|) = Pois(\lambda\frac{4}{3}\pi x^3)$$

And thus,

$$\begin{aligned}
F_R(x) &= Pr\{R \leq x\} && \text{by definition of c.d.f} \\
&= 1 - Pr\{R > x\} && \text{complementary event} \\
&= 1 - Pr\{N(A) = 0\} && \text{as explained before} \\
&= 1 - \left[e^{-\lambda\frac{4}{3}\pi x^3}\frac{(\lambda\frac{4}{3}\pi x^3)^0}{0!}\right] && \text{Poisson p.m.f} \\
&= 1 - e^{-\lambda\frac{4}{3}\pi x^3} && \text{simplifying}
\end{aligned}$$

Finally, take the derivative and find the p.d.f:

$$f_R(x) = \frac{d}{dx}F_R(x) = \frac{d}{dx}\left[1 - e^{-\lambda\frac{4}{3}\pi x^3}\right] = -(-\lambda\frac{4}{3}\pi 3x^2)e^{-\lambda\frac{4}{3}\pi x^3} = \boxed{(4\lambda\pi x^2)e^{\frac{-4\lambda\pi x^3}{3}}}.$$