M463 Homework 5

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(1.6) #6 Suppose you roll a fair six-sided die repeatedly until the first time you roll a number that you have rolled before.

a) For each $r = 1, 2, \cdots$ calculate the probability p_r that you roll exactly r times.

Solution: First note that for the values $r = 1$ or $r \geq 8$ the probability is 0. This is because, in the first case, if you only roll once then there is no possible way to get a repetition. In the latter, it is not possible to have to roll the die 8 or more times to get a repetition since there are only 6 possibilities for the die. In other words, the maximum value for which we have a positive probability is seven which occurs in the case in which we get all different numbers in the previous six rolls.

To compute the probabilities, fix a number of rolls, say $r = 4$. Then, you know that in the last roll you have to have a repetition and there are 6 choices for this number. You still have three positions to fill, but one of these positions is predetermined by the last roll so you have $5 \cdot 4$ ways of doing so. Also, you want to permute the way the rolls appear since it might be that the repeated number is equal to the first or second or third roll, so 3 ways of permuting these. Finally note that $|\Omega| = 6^3$, since these are all the E.L.O for tossing a die three times. In summary, for this case we have:

$$P(\text{exactly 4 rolls}) = \frac{\#E}{\#\Omega} = \frac{(6 \cdot 5 \cdot 4) \cdot 3}{6^3}$$

Extending this argument to $r = 2, 3, \cdots 7$, we obtain the following formula valid only when $1 \leq r \leq 7$:

$$\boxed{P_r = \frac{(6)_{r-1} \cdot (r-1)}{6^{r-1}}}$$

where $(6)_{r-1}$ is the permutation function. The following table summarizes the probabilities.

r	P_r
1	0
2	1/6
3	5/18
4	5/18
5	5/27
6	25/324
7	5/324

Also, $P_r = 0$ for $r \geq 8$.

b) Without calculation, write down the value of $p_1 + p_2 + \cdots + p_{10}$. Explain.

Solution: Since $p_1 + p_2 + \cdots + p_{10}$ cover all possibilities, i.e., you either get a repeated number on the die before rolls 1 through 10 (in fact, you have to get a repetition from rolls seven and up) the event $E =$ "roll exacty i times" for $i = 1, \cdots 10$ is such that $P(E) = p_1 + p_2 + \cdots + p_{10} = \boxed{1}$

c) Check that your calculated values of p_r have this value for their sum.

Solution: Using the calculated values:

$$\sum_{i=1}^{10} P_r = P_1 + P_2 + \cdots + P_{10} = 0 + \frac{1}{6} + \frac{5}{18} + \frac{5}{18} + \frac{5}{27} + \frac{25}{324} + \frac{5}{324} + 0 + 0 + 0 = \boxed{1}$$