M463 Homework 14

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An item is selected randomly from a collection labeled $1, 2, \cdots, n$. Denote its label by X. Now select an integer Y uniformly at random from $\{1, 2, \cdots, n\}$. Find:

a) $E(Y)$;

Solution: Using the double expectation formula: $E(Y) = E(E(Y|X))$. Note that $Y|X$ is distributed uniformly on $\{1, \cdots, X\}$, where we think of X here as a fixed value. Therefore, $E(Y|X) = \dfrac{X+1}{2}$, so,

$$E(Y) = E(E(Y|X)) = E\left(\frac{X+1}{2}\right) = \frac{E(X)+1}{2} = \frac{\frac{n+1}{2}+1}{2} = \frac{\frac{n+3}{2}}{2} = \boxed{\frac{n+3}{4}}$$

b) $E(Y^2)$;

Solution: As in part a): $E(Y^2) = E(E(Y^2|X))$. Note that second moment of Y is the same as the second moment of a uniform distribution on $\{1, \cdots, X\}$, where, again, X is fixed. We derived in class the second moment of a uniform distribution. Apply this to: $E(Y^2|X) = \dfrac{2X^2+3X+1}{6}$, so,

$$E(Y^2) = E\left(\frac{2X^2+3X+1}{6}\right) = \frac{1}{6}\left[2E(X^2)+3E(X)+1\right] = \frac{1}{6}\left[2\left(\frac{2n^2+3n+1}{6}\right)+3\left(\frac{n+1}{2}\right)+1\right]$$

$$= \frac{1}{6}\left[\frac{4n^2+6n+2+9n+9+6}{6}\right] = \boxed{\frac{4n^2+15n+17}{36}}$$

c) $SD(Y)$;

Solution: $Var(Y) = E(Y^2) - E(Y)^2 = \dfrac{4n^2+15n+17}{36} - \left(\dfrac{n+3}{4}\right)^2 = \dfrac{4n^2+15n+17}{36} - \dfrac{n^2+6n+9}{16}$

$= \dfrac{16n^2+60n+68-9n^2-54n-81}{144} = \dfrac{7n^2+6n-13}{144}$. Therefore, $S.D.(Y) = \boxed{\sqrt{\dfrac{7n^2+6n-13}{144}}}$

d) $P(X+Y=2)$.

Solution: Since both X and Y take integers values starting from 1, the only way the sum can be equal to 2 is if both X and Y are 1. Hence,

$$P(X+Y=2) = P(X=1, Y=1) = P(Y=1|X=1)P(X=1) = 1 \cdot \frac{1}{n} = \boxed{\frac{1}{n}}$$

Note that $P(Y=1|X=1) = 1$ since if $X=1$ then Y only takes the value 1.