M463 Homework 11

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Let A_1, A_2, and A_3 be events with probabilities $\frac{1}{5}, \frac{1}{4}$, and $\frac{1}{3}$, respectively. Let N be the number of these events that occur.

a) Write down a formula for N in terms of indicators.
Solution: Let I_i be defined as:

$$I_i = \begin{cases} 1 \text{ if event } A_i \text{ occurs} \\ 0 \text{ otherwise} \end{cases}$$

for $i = 1, 2, 3$. Then

$$\boxed{N = I_1 + I_2 + I_3}$$

b) Find $E(N)$.
Solution: Since I_i are indicators R.V.s, we have that $E(I_1) = \frac{1}{5}, E(I_2) = \frac{1}{4}$, and $E(I_3) = \frac{1}{3}$.
By linearity of expected value: $E(N) = E(I_1 + I_2 + I_3) = E(I_1) + E(I_2) + E(I_3) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} = \frac{12+15+20}{60} = \boxed{\frac{47}{60}}$

In each of the following cases, calculate $Var(N)$:

c) A_1, A_2, A_3 are disjoint;
Solution:

$$\begin{aligned} Var(N) &= Var(I_1 + I_2 + I_3) && \text{By Def. of } N \\ &= E[(I_1 + I_2 + I_3)^2] - [E(I_1 + I_2 + I_3)]^2 && \text{Computational formular for } Var \end{aligned}$$

The second term of this expression is $[E(I_1 + I_2 + I_3)]^2 = E(N)^2 = \left(\frac{47}{60}\right)^2 = \frac{2209}{3600}$.

The first term is slightly more complicated to compute:

$$\begin{aligned} E[(I_1 + I_2 + I_3)^2] &= E[I_1^2 + I_2^2 + I_3^2 + 2I_1I_2 + 2I_1I_3 + 2I_2I_3] && \text{Squaring inside } E \\ &= E(I_1^2) + E(I_2^2) + E(I_3^2) + 2E(I_1I_2) + 2E(I_1I_3) + 2E(I_2I_3)] && \text{Linearity of } E \\ &= E(I_1) + E(I_2) + E(I_3) + 2E(I_1I_2) + 2E(I_1I_3) + 2E(I_2I_3)] && \text{Idempotence of } I_i \end{aligned}$$

But, $I_iI_j = I_{i \cap j}$, so $P(I_{i \cap j} = 1) = 0$ since events are mutually disjoint. So, $E(I_iI_j) = 0$.

Hence, $E[(I_1 + I_2 + I_3)^2] = E(I_1) + E(I_2) + E(I_3) = E(N) = \frac{47}{60}$. Combining these two expressions:

$$Var(N) = E(N) - E(N)^2 = \frac{47}{60} - \frac{2209}{3600} = \boxed{\frac{611}{3600}}$$

d) they are independent;
Solution: Note that $Var(I_1) = pq = \frac{1}{5} \cdot \frac{4}{5} = \frac{4}{25}$. Likewise, $Var(I_2) = \frac{3}{16}$, and $Var(I_3) = \frac{2}{9}$.

By independence, $Var(N) = Var(I_1 + I_2 + I_3) = Var(I_1) + Var(I_2) + Var(I_3) = \frac{4}{25} + \frac{3}{16} + \frac{2}{9} = \boxed{\frac{2051}{3600}}$

c) $A_1 \subset A_2 \subset A_3$.
Solution: We proceed as we did for part c).

$$\begin{aligned} Var(N) &= Var(I_1 + I_2 + I_3) && \text{By Def. of } N \\ &= E[(I_1 + I_2 + I_3)^2] - [E(I_1 + I_2 + I_3)]^2 && \text{Computational formular for } Var \end{aligned}$$

The second term was previously computed in c).
For the first term:

$$E[(I_1 + I_2 + I_3)^2] = E(I_1) + E(I_2) + E(I_3) + 2E(I_1 I_2) + 2E(I_1 I_3) + 2E(I_2 I_3) \quad \text{Previous calculation}$$

But, $I_1 I_2 = I_1$ since $A_1 \subset A_2$. Likewise, $I_1 I_3 = I_1$ since $A_1 \subset A_3$ and $I_2 I_3 = I_2$ since $A_2 \subset A_3$. Hence,

$$\begin{aligned} E[(I_1 + I_2 + I_3)^2] &= E(I_1) + E(I_2) + E(I_3) + 2E(I_1 I_2) + 2E(I_1 I_3) + 2E(I_2 I_3) \\ &= 5E(I_1) + 3E(I_2) + E(I_3) \\ &= \frac{5}{5} + \frac{3}{4} + \frac{1}{3} = \frac{60 + 45 + 20}{60} = \frac{125}{60} = \frac{25}{12} \end{aligned}$$

Finally,

$$Var(N) = E[(I_1 + I_2 + I_3)^2] - E(N)^2 = \frac{25}{12} - \frac{2209}{3600} = \boxed{\frac{5291}{3600}}$$