M451/551 Quiz 6

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You do not need to simplify numerical expressions.

1. Find the no-arbitrage cost of a European (K, T) call option on a security that, at times t_{d_i} $(i = 1, 2)$, pays $fS(t_{d_i})$ as dividends, where $t_{d_1} < t_{d_2} < T$. Express your answer in terms of the standard Black-Scholes function $C(S, K, T, r, \sigma)$.

The value of the investment $M(y)$ is

$$\begin{cases} S(y) & \text{if } y < t_{d_1} \\ \frac{1}{1-f} S(y) & \text{if } t_{d_1} \leq y < t_{d_2} \\ \frac{1}{(1-f)^2} S(y) & \text{if } y > t_{d_2} \end{cases}$$

since we reinvest immediately

So, if $t > t_{d_2}$ $\Rightarrow \frac{S(t)}{S(0)} = \frac{1}{(1-f)^2} \frac{M(t)}{M(0)}$ but $\frac{S(t)}{S(0)}$ follows G.B.M, so

$\frac{S(t)}{S(0)} = e^{W(t)} (1-f)^2 \Rightarrow S(t) = S(0)(1-f)^2 e^{W(t)}$, where $W(t)$ is a Normal R.V.

therefore, the cost is

$C = e^{-rt} E_{\text{(no arbitrage)}}[\text{return}] = e^{-rT}[(S(t) - K)^+] = e^{-rT}\{(S(0)(1-f)^2 e^{W(t)} - K)^+]$

But this is just B.S. so the cost is

$$\boxed{C(S(0)(1-f)^2, K, T, r, \sigma)}$$

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(Problem #2 is on the other side.)

2. Consider a European (K, T) call option whose return at expiration time is capped by the amount B. That is, the payoff at T is $\min((S(T) - K)^+, B)$. Explain how you can use the BlackScholes formula to find the no-arbitrage cost of this option.

Consider two investments:

$I_1 = (K, T)$ - call option on same security as capped option

$I_2 = (K+B, T)$ - call option " " " " " "

The pay off of each is

$$\text{payoff } I_1 = \begin{cases} S(t) - K & \text{if } S(t) > K \\ 0 & \text{o/w} \end{cases}$$

$$\text{pay off } I_2 = \begin{cases} S(t) - (K+B) & \text{if } S(t) > K+B \\ 0 & \text{o/w} \end{cases}$$

The pay off of the difference is exactly the payoff of the capped option since: $I_1 - I_2$

Here we assume $B \geq 0$.

$$\text{payoff } I_1 - \text{payoff } I_2 = \begin{cases} S(t) - K - 0 = S(t) - K \ldots \text{if } \begin{cases} S(t) > K \\ S(t) < K+B \end{cases} \\ S(t) - K - [S(t) - (K+B)] = B \ldots \text{if } \begin{cases} S(t) > K \\ S(t) > K+B \end{cases} \\ 0 - 0 = 0 \ldots \ldots \text{o/w } \{ S(t) < K \Rightarrow S(t) < K+B \end{cases}$$

By the law of one price, since investment $I_1 - I_2$ has the same payoff as the capped option, their cost must be the same. But we know cost of I_1 is, by B-S, $C(S(0), K, T, r, \sigma)$

AND cost of I_2 is, by B-S, $C(S(0), K+B, T, r, \sigma)$

Hence, the cost of capped option, call it C is:

+10

$$\boxed{C = C(S(0), K, T, r, \sigma) - C(S(0), K+B, T, r, \sigma)}$$