M451/551 Exam 2

April 16, 2015

Name: Enrique Areyan

INSTRUCTIONS: Please make sure your exam has 7 pages, in addition to this cover page. **You must justify your solutions to receive credit.** Please try to fit your solutions into the space provided. If you do need extra space, please write "continued on the back," and continue on the back of the same sheet. Also, be sure to indicate your final answer to each problem clearly.

Do not write below this line. For graders use:

1.	5	5.	15
2.	15	6.	14
3.	15	7.	15
4.	9	Sum	88

Formulae:

Black-Scholes:

$$C(S,K,T,r,\sigma) = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}) \quad \text{where} \quad \omega = \frac{(r+\sigma^2/2)T - \log\frac{K}{S}}{\sigma\sqrt{T}}$$

PDF of Standard Normal Random Variable $Z_{0,1}$:

$$\Phi'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Put-Call Parity:

$$S + P - C = Ke^{-rT}$$

$\boxed{\text{Just a note:}}$ Itô's lemma wasn't clear at all from class or hw.

Problem 1. (10 PTS.) *Itô's Lemma states that if X_t satisfies $dX_t = \mu(t)dt + \sigma(t)dW_t$ where W_t is the Brownian Motion with drift 0 and volatility 1, then for any function $f(x,t)$ we have,*

$$df(X_t,t) = \frac{df}{dt}(X_t,t)dt + \frac{df}{dx}(X_t,t)dX_t + \sigma^2(t)/2\frac{d^2f}{dx^2}(X_t,t)dt.$$

Apply this to find $df(X_t,t)$ for the function $f(x,t) = e^{xt}$ in terms of dt and dW_t. (I.e. your answer needs to have the form $df(X_t,t) = a(X_t,t)dt + b(X_t,t)dW_t$ for explicit functions $a(x,t)$ and $b(x,t)$ which can also involve $\mu(t)$ and $\sigma(t)$.)

Want to find $df(X_t,t)$ for the function $f(x,t)=e^{xt}$.

Apply Lemma by pieces: Considering: $(X_t,t)=e^{xt}$

$$\frac{df}{dt}e^{xt}dt = xe^{xt}dT$$

$$\frac{df}{dx}e^{xt}dX_t = te^{xt}dX_t$$

$$\frac{\sigma^2(t)}{2}\frac{d^2f}{dx^2}\cdot e^{xt}dt = \frac{\sigma^2}{2}t^2e^{xt}dT$$

$$= x_t e^{xt}dt + te^{xt}dX_t + \frac{\sigma^2(t)}{2}t^2e^{xt}dt$$

$$= e^{xt}\left(x + \frac{\sigma^2(t)}{2}t^2\right)dt + te^{xt}dX_t$$

At this point I'm not sure how to proceed. However, I do recall from class discussion that the answer here should be B-S.

+5

Problem 2. (15 PTS.) *Find the no-arbitrage cost of a European (K, T) call option on a security that, at times t_i $(i = 1, 2)$, pays $f_i S(t_i)$ as dividends, where $t_1 < t_2 < T$ and $0 < f_1 < f_2 < 1$.*

the cost is $C((1-f_1)(1-f_2)S(0), K, T, r, \sigma)$.

Following the Model for options pricing on dividends, we know that if we get a single dividen f_1 as a fraction of the stock's price, then the cost of an optio is $C((1-f_1)S(0), K, T, r, \sigma)$.

The reason is that, the value of our portfolio $M(y)$ is given by

$$M(y) = \begin{cases} S(y) & \text{if } t_1 < t \\ (1-f_1)S(y) & \text{if } t > t_1 \end{cases}$$

Assuming M follows G.B.M under risk neutral prob., we get that $S(t) = S(0)(1-f_1)e^{W}$, where $W \sim \text{Normal}\left(\left(r - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$

So the cost is $C = e^{-rT} E[\text{return}] = e^{-rT}\left[\left(S(0)(1-f_1)e^{W} - K\right)^+\right]$

$= C(S(0)(1-f_1), K, T, r, \sigma)$.

Now, follow this same model, but consider a second dividend that multiplies the initial price of $S(0)(1-f_1)$ to get:

$$\boxed{\text{Cost} = C((1-f_1)(1-f_2)S(0), K, T, r, \sigma)}$$

15

Problem 3. (15 PTS.) *The current price of a security is $S(0)$. Consider an investment whose cost is Q and whose payoff at time T is, for a specified choice of R satisfying $0 < R < e^{rT} - 1$, given by*

$$return = \begin{cases} (1+R)S(0) + \alpha(S(T) - (1+R)S(0)) & \text{if } S(T) \geq (1+R)S(0), \\ (1+R)S(0) & \text{if } S(T) \leq (1+R)S(0). \end{cases}$$

Determine the value of α if this investment (whose payoff is both uncapped and always greater than the initial cost of the investment) is not to give rise to an arbitrage.

The return of this investment is given by:

$$E\left[(1+R)S(0) + \alpha[S(T) - (1+R)S(0)]^+\right] =$$

$$(1+R)S(0) + \alpha E[(S(T) - (1+R)S(0))^+] =$$

Under the risk-neutral G.M. we have that

$$= (1+R)S(0) + \alpha e^{rT} C(S(0), (1+R)S(0), T, r, 6)$$

So this is the payoff at time T (this is why I have this term).

But, for there to be no arbitrage, cost of investment having equal payoff should be equal, I.e., sell the investment at time 0 and get $e^{rT}Q$ at time 1. then

$$e^{rT}Q = (1+R)S(0) + \alpha e^{rT} C(S(0), (1+R)S(0), T, r, 6).$$

Solve for α :

$$\boxed{\frac{e^{rT}Q - (1+R)S(0)}{e^{rT} C(S(0), (1+R)S(0), T, r, 6)} = \alpha}$$

+15

Problem 4. (15 PTS.) *Consider a European (K, T) put option whose return at expiration time T is capped by the amount B. That is, the payoff at T is*

$$\min((K - S(T))^+, B).$$

Explain how you can use the BlackScholes formula to find the no-arbitrage cost of this option. Hint: Start by expressing the payoff in terms of the payoffs from two plain (uncapped) put options.

Consider the following 2 investments:

I_1 : Call option (K, T)

I_2 : call option $(K+B, T)$.

9

the payoff of each of these is:

$$\text{Payoff } I_1 = \begin{cases} S(t) - K & \text{if } S(t) > K \\ 0 & \text{o/w} \end{cases}$$

$$\text{payoff } I_2 = \begin{cases} S(t) - (K+B) & \text{if } S(t) > K+B \\ 0 & \text{o/w} \end{cases}$$

the payoff of the difference is the same as the payoff of of a capped call option:

$$\text{payoff capped call option} = \begin{cases} S(t) - K & \text{if } S(t) > K \text{ and } K - S(t) \leq B \\ B & \text{if } K - S(t) > B \\ 0 & \text{o/w} \end{cases}$$

write put payoff

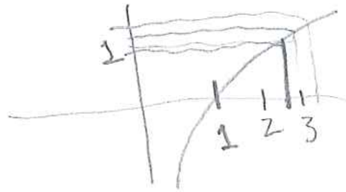
For there to be no arbitrage the costs of the capped option should be the cost of the difference $I_1 - I_2$. the cost of I_1 is $C(S(0), K, T, r\, \sigma)$ and cost of I_2 is $C(S(0), K+B, T, r, \sigma)$, so the cost of the capped call option is $C_c = C(S(0), K, T, r, \sigma) - C(S(0), K+B, T, r, \sigma)$

Finally, use put-call parity option to get the price of a put option:

Cost of a put for $I_1 = Ke^{-rT} + C(S(0), K, T, r, \sigma) - S$ ①

Cost of a put for $I_2 = Ke^{-rt} + C(S(0), K+B, T, r, \sigma) - S$ ②

Cost of put capped option = ① - ② = Turns out to be the same as before.



Problem 5. (15 PTS.) *The utility function of an investor is $u(x) = 1 - 1/x$. The investor must choose one of two investments. If his fortune after investment 1 is a random variable with density function $f_1(x) = x/4$, $1 < x < 3$, and his fortune after investment 2 is a random variable with density function $f_2(x) = 1/2$, $1 < x < 3$, which investment should he choose?*

Let X be investment 1. pdf of X is $f_1(x) = \frac{x}{4}$, $1 < x < 3$

Let Y be investment 2. pdf of Y is $f_2(y) = 1/2$, $1 < x < 3$

The investor should choose $\max\{E[u(X)], E[u(Y)]\}$.

Let us compute each of this:

$$E[u(X)] = \int_1^3 \left(1 - \frac{1}{x}\right) \cdot \frac{x}{4}\, dx = \int_1^3 \frac{x}{4} - \frac{1}{4}\, dx = \frac{1}{4}\int_1^3 x - 1\, dx$$

$$= \frac{1}{4}\left[\int_1^3 x\, dx - 2\right] = \frac{1}{4}\left\{\left[\frac{x^2}{2}\right]_1^3 - 2\right\} = \frac{1}{4}\left\{\left(\frac{9}{2} - \frac{1}{2}\right) - 2\right\} = \frac{1}{4}\left[4 - 2\right] = \frac{1}{4} \cdot 2 = \boxed{\frac{1}{2}}$$

$$E[u(Y)] = \int_1^3 \left(1 - \frac{1}{x}\right)\frac{1}{2}\, dx = \frac{1}{2}\int_1^3 1 - \frac{1}{x}\, dx = \frac{1}{2}\left\{2 - \int_1^3 \frac{1}{x}\, dx\right\}$$

$$= \frac{1}{2}\left\{2 - \ln(x)\Big|_1^3\right\} = \frac{1}{2}\left\{2 - \ln(3) + \cancel{\ln(1)}^{0}\right\} = \frac{1}{2}\left\{2 - \ln(3)\right\}$$

Since $1 < \ln(3) < 2 \Rightarrow 2 - \ln(3) < 1$, Hence:

15

$$\frac{1}{2}(2 - \ln(3)) < \frac{1}{2}$$

$$\begin{array}{cc} \| & \| \\ E[u(Y)] & E[u(X)] \end{array}$$

$\boxed{\text{So choose investment 1.}}$

Note $X \sim \text{Uniform}(0,1)$

$E[X] = \frac{1}{2}$; $E[X^2] = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

Problem 6. (15 PTS.) *Suppose your utility function is $u(x) = 2-(x-1)^2$, and that starting with initial wealth $1 < W_0 < 2$ your total wealth from all investments after one period is $W = \alpha W_0 X + (1-\alpha)W_0$ where X is a random variable with uniform density $f_X(x) = 1$ for $0 < x < 1$. Find the $\alpha \in [0,1]$ that maximizes $E[u(W)]$.*

$$E[u(W)] = E[2-(\alpha W_0 X + (1-\alpha)W_0 - 1)^2]$$
$$= E[2-(\alpha W_0 X + W_0 - \alpha W_0 - 1)^2]$$
$$= E[2-(W_0(\alpha X + 1 - \alpha) - 1)^2]$$
$$= E[2-(W_0^2(\alpha X + 1-\alpha)^2 - 2W_0(\alpha X + 1 - \alpha) + 1)]$$
$$= 2 - \{W_0^2 E[(\alpha(X-1)+1)^2] - 2W_0 E[\alpha X + 1 - \alpha] + 1\}$$
$$= 2 - \{W_0^2 E[\alpha^2(X-1)^2 + 2\alpha(X-1) + 1] - 2W_0 E[\alpha X + 1 - \alpha] + 1\}$$
$$= 2 - \{W_0^2[\alpha^2 E[(X-1)^2] + 2\alpha E[X-1] + 1] - 2W_0[\alpha E[X] + 1 - \alpha + 1\}$$
$$= 2 - \{W_0^2[\alpha^2 E[X^2 - 2X + 1] + 2\alpha[\tfrac{1}{2} - 1] + 1] - 2W_0[\alpha \tfrac{1}{2} + 2 - \alpha]\}$$
$$= 2 - \{W_0^2[\alpha^2[\tfrac{1}{3} - \cancel{X} + \cancel{1}] + 2\alpha[-\tfrac{1}{2}] + 1 - 2W_0[2 - \tfrac{1}{2}\alpha]\}$$
$$= 2 - \{\tfrac{1}{3}W_0^2\alpha^2 - \alpha W_0^2 + 1 - 4W_0 + W_0\alpha\}$$
$$= 1 - \tfrac{1}{3}W_0^2\alpha^2 + \alpha + 4W_0 - W_0\alpha = f(\alpha).$$ So, MAximize $f(\alpha)$, $\alpha \in [0,1]$

$$\frac{d}{d\alpha} f(\alpha) = -\frac{2}{3}W_0^2\alpha + W_0^2 - W_0 = 0 \Rightarrow \frac{2}{3}W_0^2\alpha = 1 - W_0 \Rightarrow \boxed{\alpha^* = \frac{3}{2}\,\frac{1-W_0}{W_0^2}}$$

Second derivative shows this is a max: $\frac{d^2}{d\alpha^2} f = -\frac{2}{3}W_0^2 < 0$, for any W_0.

If $1 < W_0 < 2$ then $\alpha^* < 0$ (since numerator is $1-W_0$), in that case the optimal would be $\alpha^* = 0$,

since if $1 < W_0 < 2$ $\alpha^* < 0$ an a risk-averse investor would invest zero in this

the actual optimal is $\boxed{\alpha^* = 0}$, i.e., optimal at end point.

+14

Problem 7. (15 PTS.) *We showed that for any random variable X, $E[X] \geq_{icv} X$. By contrast, find:*

(a) *An example of an $X \geq 0$ such that $E[X] \not\geq_{lr} X$*

(b) *An example of an $X \geq 0$ such that $E[X] \not\geq_{st} X$*

(Hint: $E[X]$ is a constant with density a single Dirac distribution at $E[X]$ and now use the definitions of the the dominance $\geq_{lr}$ and $\geq_{st}$.)

15

(b) By definition $X \geq_{s.t.} Y$ if $P(X>t) \geq P(Y>t)$ for all t.

let $X = \begin{cases} 0 & \text{with prob. } 1/2 \\ 1 & \text{'' '' } 1/2 \end{cases} \Rightarrow E[X] = \frac{1}{2}$, but consider $t =$

$P(E[X] > \frac{3}{4}) = 0 < P(X > \frac{3}{4}) = \frac{1}{2}$,

$\Downarrow$ since $E[X]$ is always $\frac{1}{2}$ $\quad$ $\Downarrow$ since X could be 1 with 1/2 prob.

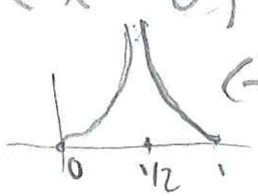
Hence $E[X] \not\geq_{st} X$.

(a) By definition $X \geq_{lr} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is nondecreasing in t for all regions where either X, Y is defined. In case of a discrete prob. dist.

$\frac{P(X=k)}{P(Y=k)}$ is nondecreasing in k, again where either $P(X=k)$ or $P(Y=k)$ is nonzero.

Using same X as before

$\frac{P(E[X]=0)}{P(X=0)} = 0$; $\frac{P(E[X]=\frac{1}{2})}{P(X=\frac{1}{2})} = \infty$; $\frac{P(E[X]=1)}{P(X=1)} = 0$



this is obviously not non decreasing (look at picture)

So this same X provides an example where $X \geq 0$, $E[X] \not\geq_{lr} X$