CHAPTER 5:

Ex (5.1): pay 10 to buy a European ($K=100, t=2$) call option. Assume continuously compounded nominal annual interest rate of 6%. Find the present value of the return from investment if
(a) $S(2)=110$ and (b) $S(2)=98$.

Since this is a call option, the future value of the call is

$$\text{Value (future)} = \begin{cases} S(t)-K & \text{if } S(t)>K \\ 0 & \text{if } S(t)\leq K \end{cases}$$

In case (a), the future value is $S(2)-K = 110-100=10$, since $S(2)>K$. So the present value is $10e^{-0.06*2} = 10e^{-0.12} = 8.869204$, from which we need to substract the cost to get P.V. return:

$$10e^{-0.12}-10 = \boxed{-1.13080\$}$$

In case (b), the future value is 0, since $S(2)=98<100=K$.
So the present value of return is $\boxed{-10\$}$

Ex (5.2): pay 5 to buy a European ($K=100, t=1/2$) put option. Assume monthy compounded nominal annual interest rate of 6%. Find the present value of the return from investment if
(a) $S(1/2)=102$ and (b) $S(1/2)=98$.

Since this is a put option, the future value of the put is

$$\text{value (future)} = \begin{cases} K-S(t) & \text{if } S(t)<K \\ 0 & \text{if } S(t)\geq K \end{cases}$$

In case (a), the future value is 0, since $S(1/2)=102>K=100$
So the present value of return is $\boxed{-5\$}$

In case (b), the future value is $K - S(1/2) = 100 - 98 = 2$, since $S(1/2) < K$. So the present value is $2 * (1.005)^{-6} = 1.94103$, from which we need to substract the cost to get P.V. of return:

$$2 * (1.005)^{-6} - 5 = \boxed{-3.05896 \$}$$

Ex (5.3): Since $K < \min s_i$, it follows that the call option will be exercised. To find the cost of an option we could use the law of one price. For that we need two different ways of getting the same payouts and further assume no arbitrage.

So, assuming no arbitrage, consider the following 2 investments, both of which yield the security at time 1:

1) Just buy the security at time 0 for its initial price S.

2) buy a call option at cost C at time 0 and exercise it at time 1. Exercising it costs K at time 1, so the present cost of this investment is, assuming compounding interest rate r,

$$C + Ke^{-r}.$$

($\Rightarrow$) Since investments (1) and (2) have the same payout (1 stock at time 1), and we assume no arbitrage, it follows that they should have the same cost: $S = C + Ke^{-r}$, from which we get the cost of the option:

$$\boxed{C = S - Ke^{-r}}$$

Note that this formula makes sense: If interest rates are high ($r \to \infty$) then $C = S$, the cost is just its present value.
If interest rates are low, say $r = 0$, then $C = S - K$, so the cost is just the current price minus the strike value.

Ex 5.8: Let P be the price of a put option to sell a security, whose present price is S, for the amount K. Let us argue that:

$$P \geq Ke^{-rt} - S, \quad t = \text{exercise time}, \quad r = \text{interest rate}.$$

By assuming that $P < Ke^{-rt} - S$. (*)

Then, by the put call option parity formula:

$$S + P - C = Ke^{-rt}$$

from which follows: $S = Ke^{-rt} - P + C$, Subtitute this in (*)

$$P < Ke^{-rt} - S = \cancel{Ke^{-rt}} - (\cancel{Ke^{-rt}} - P + C) = P - C$$

$\Rightarrow$ $P < P - C$, which is a contradiction provided $C \geq 0$

Therefore, $\boxed{P \geq Ke^{-rt} - S}$.

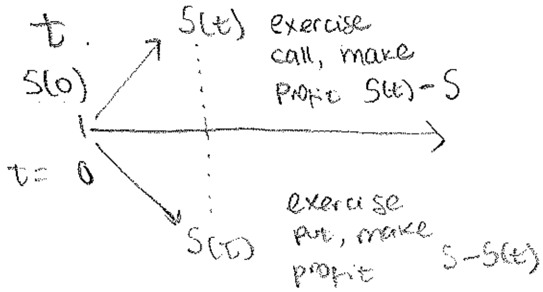
Ex 5.18: Consider the strategy: buy both a European put and a European call on the same security with both options expiring in three months, and both having a strike price equal to the present price of the security.

(a) Under what conditions would such an investment strategy seem reasonable?

Since the strategy relies on the price of the security changing enough in either one direction so that excercising either the call or the put yields a profit, we would need to have a security with a high volatility. That way we have a higher confidence in the price of the security changing with respect to its original price.

(b) Plot the return at time $t=1/4$ from this strategy as a function of the price of the security at that time.

First we would need to derive the return from this strategy at time t.

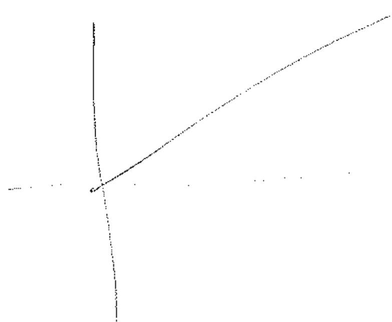


$$R = e^{-rt}|S(t) - S| - (P + C)$$

We make money so as long as $S(t) \neq S$. Suppose $S(t) > S$. (the case $S(t) < S$ is symmetrical and if $S(t) = S$ then $R = -(P+C) < 0$). Then,

$$R = e^{-rt}(\underbrace{S(t)} - S) - (P + C)$$

$$e^{-r(t-\frac{t}{1})x}$$



Ex (5.24): Let $P(K,t)$ denote the cost of a European put option with strike K an expiration time t. Prove that $P(K,t)$ is convex in K for fixed t.

Pf: this result follows from the previously shown convexity of $C(K,t)$ for a fixed t and the put call option parity formula:

$$S + P(K,t) - C(K,t) = Ke^{-rt}$$

$$\Rightarrow \quad C(K,t) = S + P(K,t) - Ke^{-rt} \quad (*)$$

Convexity of $C(K,t)$, for fixed t, means: $\forall K_1, K_2$:

$$\lambda C(K_1,t) + (1-\lambda) C(K_2,t) \geq C(\lambda K_1 + (1-\lambda) K_2, t)$$

Start from the Left-hand side, and use $(*)$:

$$\lambda C(K_1,t) + (1-\lambda) C(K_2,t) = \lambda[S + P(K_1,t) - K_1 e^{-rt}] + (1-\lambda)[S + P(K_2,t) - K_2 e^{-rt}]$$

$$= \lambda P(K_1,t) + (1-\lambda) P(K_2,t) + \underbrace{\lambda S - \lambda K_1 e^{-rt} + (1-\lambda)(S - K_2 e^{-rt})}_{2} \geq \text{(by hypothesis)}$$

$$C(\lambda K_1 + (1-\lambda) K_2, t) = S + P(\lambda K_1 + (1-\lambda) K_2, t) - (\lambda K_1 + (1-\lambda) K_2) e^{-rt}$$

$$= P(\lambda K_1 + (1-\lambda) K_2, t) + \underbrace{S - (\lambda K_1 + (1-\lambda) K_2) e^{-rt}}_{1}$$

We are almost there. We just need to show $1 \overset{?}{=} 2$:

$$2 = \lambda S - \lambda K_1 e^{-rt} + (1-\lambda)(S - K_2 e^{-rt})$$

$$= \cancel{\lambda S} - \lambda K_1 e^{-rt} + S - K_2 e^{-rt} - \cancel{\lambda S} + \lambda K_2 e^{-rt}$$

$$= S - \lambda K_1 e^{-rt} + K_2(\lambda e^{-rt} - e^{-rt})$$

$$= S - \lambda K_1 e^{-rt} + K_2(e^{-rt}(\lambda - 1))$$

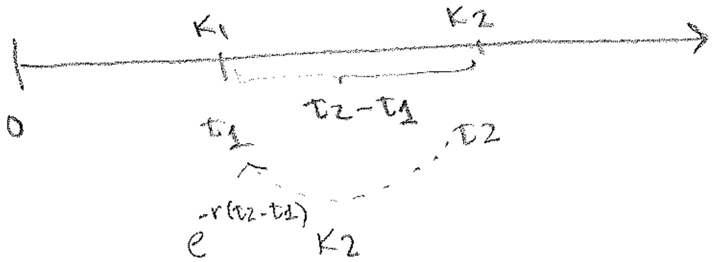
$$= S - (\lambda K_1 + (1-\lambda) K_2) e^{-rt}$$

$$= 1$$

It follows: $\boxed{\lambda P(K_1,t) + (1-\lambda) P(K_2,t) \geq P(\lambda K_1 + (1-\lambda) K_2, t)}$

which shows that $P(K,t)$ is convex in K for fixed t.

Ex: 5.26 : Consider a (K_1, t_1, K_2, t_2) double call option where it can be exercised either at time t_1 with strike price K_1 or at time t_2 $(t_2 > t_1)$ with strike price K_2.



Suppose $K_1 > e^{-r(t_2-t_1)} K_2 \Rightarrow -K_1 < -K_2 e^{-r(t_2-t_1)}$ (*)

Let $S(t_1)$ be the price of the security at time t_1. Then, the return on exercising the option at t_1, measured in time t_1 is:

$$S(t_1) - K_1$$

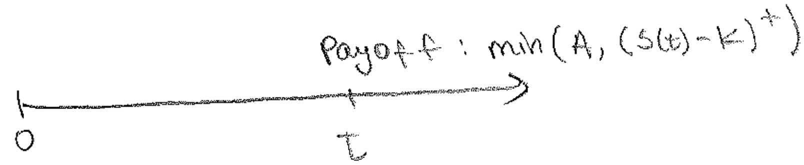
AND the return on exercising the option at t_2, measured in time t_1 is:

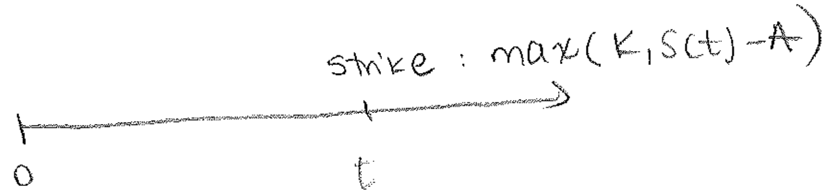
$$S(t_1) - K_2 e^{-r(t_2-t_1)}$$

But by (*), it follows: $\boxed{S(t_1) - K_1 < S(t_1) - K_2 e^{-r(t_2-t_1)}}$ so the higher return is from waiting until time t_2, provided $K_1 > e^{-r(t_2-t_1)} K_2$

Ex: 5.27 : Consider a capped call option, where the return is capped at a certain specified value A, i.e., if the option has strike price K and expiration time t, the payoff at t is $\min(A, (S(t)-K)^+)$, where $S(t)$ is the price at time t. Show that an equivalent way of defining such an option is to let $\max(K, S(t)-A)$ be the strike price when the call is exercised at time t.

Sol :

1) 

2) 

Let us compare the payoff of investment (1) & (2) to determine if they are equivalent.

(i) If $\min(A, (S(t)-K)^+) = A$, then the payoffs are

(1) A (by definition)

(2) $\min(A, (S(t)-K)^+) = A \Rightarrow A \leq (S(t)-K)^+$

But A is a cap on the payoffs, so $0 \leq A \leq (S(t)-K)^+$,

So $(S(t)-K)^+ = S(t)-K$. Hence, $A \leq (S(t)-K)^+ = S(t)-K$

$\Rightarrow K \leq S(t) - A \Rightarrow \max(K, S(t)-A) = S(t)-A$.

The payoff of this option is then:

$S(t) - (S(t)-A) = A$, just like (1).

(ii) o/w, If $\min(A, (S(t)-K)^+) = (S(t)-K)^+$, then the payoffs are:

(1) $(S(t)-K)^+$

(2) $\min(A, (S(t)-K)^+) = (S(t)-K)^+$

Either we don't exercise the option ($S(t) \leq K \Rightarrow (S(t)-K)^+ = 0$ for both options) or we exercise the option so $S(t) > K$ and then $(S(t)-K)^+ = S(t)-K < A \Rightarrow S(t) - A < K$

$\Rightarrow \max(K, S(t)-A) = K$.

the payoff for (2) is then $S(t)-K$, just like (1)

In either case (i) and (ii) the payoffs are the same, so these are equivalent options.

#8. At $t=0$, the price of a certain stock is $S(0) = \$50$.
At $t=1$, the price is either $S(1) = \$80$ or $S(1) = \$30$.
A certain option contract is worth $\$10$ if the stock price is $\$80$, and is worth $\$0$ if the stock price is $\$30$.
Assuming no arbitrage opportunities, and continuously compounded interest rate of 5%, what is the price of the option at $t=0$?

Sol.:

$$\text{value} = \begin{cases} 80x + 10y & \text{if } S(1) = 80 \\ 30x & \text{if } S(1) = 30 \end{cases}$$

$$\Rightarrow \quad 80x + 10y = 30x \Rightarrow 50x = -10y \Rightarrow x = -\frac{1}{5}y \Rightarrow y = -5x$$

$$\text{cost} = 50x + Cy = 50x + C(-5x) = 50x - 5Cx.$$

$$\text{gain} = \text{value} - \text{cost} = 30x - (50x - 5Cx)e^{0.05} \quad (\text{at } t=1)$$

we want gain to be zero in order to be no arbitrage:

$$0 = 30x - (50x - 5Cx)e^{0.05}, \text{ but we want P.V.}$$

$$0 = 30x e^{-0.05} - (50x - 5Cx) = x[30e^{-0.05} - 50 + 5C]$$

$$\Rightarrow \text{since } x \neq 0 \Rightarrow 30e^{-0.05} - 50 + 5C = 0 \Rightarrow 5C = -30e^{-0.05} + 50$$

$$\Rightarrow C = \frac{50 - 30e^{-0.05}}{5} \Rightarrow \boxed{C = 10 - 6e^{-0.05}} \approx \boxed{4.29263\$}$$

#9. Suppose we are in the situation of problem 8, but a certain bank thinks that the option should be worth \$5 for some reason and they are willing to sell you options at \$5.1 and buy options from you at \$4.9. Choose a portfolio of x shares of the stock and y options that you buy or sell at time 0 that will guarantee you a net return of \$$10^6$ dollar at time 1.

Sol: By our previous analysis, the gain of the portfolio of x shares and $y=-2x$ options is

$$\text{gain} = 30x - (50x - 2cx)e^{0.05} \quad (\text{at } t=1)$$

But, the bank offers $c = \$5.1$ and we want gain $= 10^6$.

So,

$$10^6 = 30x - (50x - 2 * 5.1 * x)e^{0.05}, \Leftrightarrow \text{solve for } x:$$

$$10^6 = 30x - e^{0.05} * 50x + 10.2 * e^{0.05}x \Leftrightarrow$$

$$10^6 = x(30 - 50e^{0.05} + 10.2e^{0.05}) \Leftrightarrow$$

$$x = 10^6/(30 - e^{0.05}(50 - 10.2))$$

$$x = 10^6/(30 - 39.8e^{0.05}) \Leftrightarrow x = -84455.3$$

So, we should sell approx 84455 shares and buy $y = -2(84455)$ options to guarantee a net return of \$$10^6$.