M451 -Introduction to Mathematical Finance - Homework 1

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Chapter 1

(2) A family picnic scheduled for tomorrow will be postponed if it is either cloudy or rainy. If the probability that it will be cloudy is .40, the probability that it will be rainy is .30, and the probability that it will be both rainy and cloudy is .20, what is the probability that the picnic will not be postponed?

Solution: Let C denote the event that it is cloudy tomorrow and let R denote the event that it is rainy tomorrow. Note that the picnic will be either postponed or not, i.e., these are complementary events. Therefore:

$$\begin{aligned}
P(\text{picnic will not be postponed}) &= 1 - P(\text{picnic will be postponed}) && \text{complement rule}\\
&= 1 - P(\text{it is either cloudy or rainy}) && \text{definition of picnic postponed}\\
&= 1 - P(C \cup R) && \text{using letters for the events}\\
&= 1 - [P(C) + P(R) - P(C \cap R)] && \text{addition rule}\\
&= 1 - [.40 + .30 - .20] && \text{replacing for probabilities}\\
&= 1 - .5 && \text{arithmetic}\\
&= .5
\end{aligned}$$

Hence, the probability that the picnic will not be postponed is .5

(4) A club has 120 members, of whom 35 play chess, 58 play bridge, and 27 play both chess and bridge. If a member of the club is randomly chosen, what is the conditional probability that she:

(a) plays chess given that she plays bridge;

(b) plays bridge given that she plays chess?

Solution: Let C and B denote the event that the randomly chosen player plays chess and bridge respectively. Then:

(a)

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{\frac{27}{120}}{\frac{58}{120}} = \frac{27}{58}$$

(b)

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{27}{120}}{\frac{35}{120}} = \frac{27}{35}$$

(7) If A and B are independent, show that so are

(a) A and B^c

(b) A^c and B^c

Solution: That events A and B are independent means that $P(A \cap B) = P(A) \cdot P(B)$. Hence,

(a)

$$\begin{aligned}
P(A) \cdot P(B^c) &= P(A)[1 - P(B)] && \text{by complement rule}\\
&= P(A) - P(A)P(B) && \text{distributing}\\
&= P(A) - P(A \cap B) && \text{by hypothesis } A \text{ and } B \text{ are independent}\\
&= P(A \cap B^c) && \text{Since } P(A) = P(A \cap B) + P(A \cap B^c)
\end{aligned}$$

Therefore, $P(A) \cdot P(B^c) = P(A \cap B^c)$, which shows that A and B^c are independent.

(b)

$$\begin{aligned}
P(A^c) \cdot P(B^c) &= [1 - P(A)][1 - P(B)] && \text{by complement rule}\\
&= 1 - P(B) - P(A) + P(A)P(B) && \text{distributing}\\
&= 1 - P(B) - P(A) + P(A \cap B) && \text{by hypothesis } A \text{ and } B \text{ are independent}\\
&= 1 - [P(A) + P(B) - P(A \cap B)] && \text{rearranging terms}\\
&= 1 - P(A \cup B) && \text{addition rule}\\
&= P((A \cup B)^c) && \text{complement rule}\\
&= P(A^c \cap B^c) && \text{De Morgan's law}
\end{aligned}$$

Therefore, $P(A^c) \cdot P(B^c) = P(A^c \cap B^c)$, which shows that A^c and B^c are independent.

(8) A gambling book recommends the following strategy for the game of roulette. It recommends that the gambler bet 1 on red. If red appears (which has probability 18/38 of occurring) then the gambler should take his profit of 1 and quit. If the gambler loses this bet, he should then make a second bet of size 2 and then quit. Let X denote the gambler's winnings.

(a) Find $P\{X > 0\}$

(b) Find $E[X]$

Solution: Let us construct a table to completely describe X. Note that we assume independence of each play.

$x(\$)$	$P(X = x)$	$x \cdot P(X = x)$
0	$(1 - 18/38)^2$	0
1	$18/38$	$18/38$
2	$(1 - 18/38) \cdot 18/38$	$2 \cdot 20/38 \cdot 18/38$

Form the table we can compute what we want:

(a) $P\{X > 0\} = P(X = 1 \text{ OR } X = 2) = P(X = 1) + P(X = 2) = 18/38 + 20/38 \cdot 18/38 = 18/38 \cdot (1 + 20/38) = 18/38 \cdot 58/38 = 261/361 \approx 72.30\%$

(b) $E[X] = 0 + 18/38 + 2 \cdot 20/38 \cdot 18/38 = 76/38 = 351/361 \approx 0.9723$ \$

(9) Four buses carrying 152 students from the same school arrive at a football stadium. The buses carry (respectively) 39, 33, 46, and 34 students. One of the 152 students is randomly chosen. Let X denote the number of students who were on the bus of the selected student. One of the four bus drivers is also randomly chosen. Let Y be the number of students who were on that driver's bus. Find $E[X]$ and $E[Y]$.

Solution: Let us construct a table to completely describe X and Y.

x	$P(X = x)$	$x \cdot P(X = x)$
39	$39/152$	$39^2/152$
33	$33/152$	$33^2/152$
46	$46/152$	$46^2/152$
34	$34/152$	$34^2/152$

y	$P(Y = y)$	$y \cdot P(Y = y)$
39	$1/4$	$39/4$
33	$1/4$	$33/4$
46	$1/4$	$46/4$
34	$1/4$	$34/4$

Hence, $E[X] = \frac{39^2+33^2+46^2+34^2}{152} = \frac{2941}{76} \approx 38.70$ and $E[Y] = \frac{39+33+46+34}{4} = \frac{152}{4} = 38$

(11) Verify that

$$Var(X) = E[X^2] - (E[X])^2.$$

Hint: Starting with the definition

$$Var(X) = E[(X - E[X])^2],$$

square the expression on the right side; then use the fact that the expected value of a sum of random variables is equal to the sum of their expectations.

Solution:

$$\begin{aligned}
Var(X) &= E[(X - E[X])^2] && \text{by definition} \\
&= E[X^2 - 2XE[X] + E[X]^2] && \text{squaring} \\
&= E[X^2] - 2E[XE[X]] + E[E[X]^2] && \text{by linearity of the expected value} \\
&= E[X^2] - 2E[X]E[X] + E[X]^2 && \text{since the expectation of a constant is the constant itself} \\
&= E[X^2] - 2E[X]^2 + E[X]^2 && \text{algebra} \\
&= E[X^2] - E[X]^2 && \text{arithmetic}
\end{aligned}$$

(12) A lawyer must decide whether to charge a fixed fee of \$5,000 or take a contingency fee of \$25,000 if she wins the case (and 0 if she loses). She estimates that her probability of winning is .30. Determine the mean and standard deviation of her fee if

(a) she takes the fixed fee;

(b) she takes the contingency fee.

Solution:

(a) If she takes the fixed fee then the mean of her fee is \$5,000 and the standard deviation is 0. In this case there is no randomness since she takes the fee regardless of the outcome of the case.

(b) If she takes the contingency fee, then let X = amount in dollars of the received fee. The distribution of X is given by the following table:

x	$P(X=x)$	$x \cdot P(X=x)$
0	7/10	0
25,000	3/10	$7,500$

Therefore, the mean or expected value of her fee is $E[X] = 7,500$.

The variance is given by $\sum_{i=1}^{2} p_i \cdot (x_i - E[X])^2 = [7/10 \cdot (0 - 7,500)^2] + [3/10 \cdot (25,000 - 7,500)^2] = 131250000$.

From which we conclude that the standard deviation is $\sqrt{131250000} \approx \$11,456.44$.

(19) Can you construct a pair of random variables such that $Var(X) = Var(Y) = 1$ and $Cov(X,Y) = 2$?.

Solution:

Chapter 2

(2)

Solution: