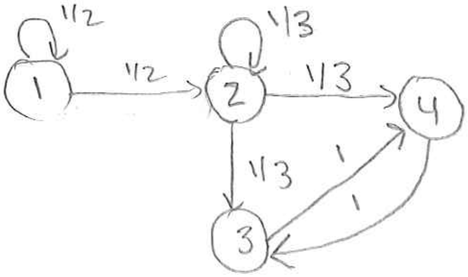
M447. - Exam 2 - Fall 2014 - Enrique Areyan (1)

60 + (6)

2. (a) the first chain diagram is:

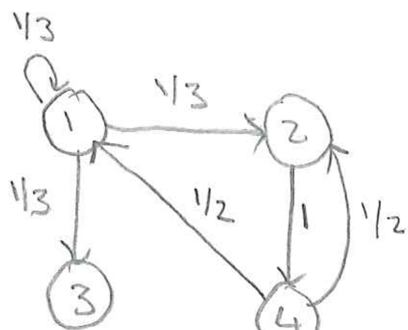
(12



We can see from the diagram that this is NOT ergodic M.C., since not all states communicate. In fact, states 1 and 3 do not communicate, since you cannot go from 3 to 1.

Since this chain is not ergodic, it follows that it is NOT regular. Finally, the chain is not absorbing since there are no absorbing states.

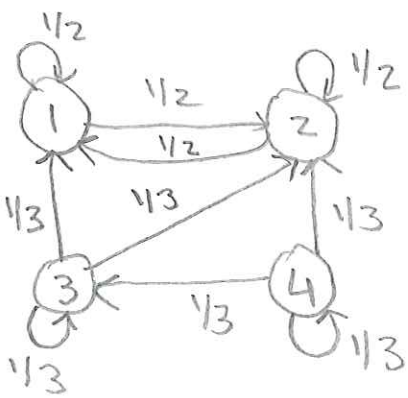
(b) the second chain diagram is:



we can see from the diagram that this is an absorbing chain, with state 3 being absorbing. It follows that the chain is not ergodic; since 3 does not communicate with any state.

Since this chain is not ergodic, it follows that it is NOT regular.

(c) the third chain diagram is:

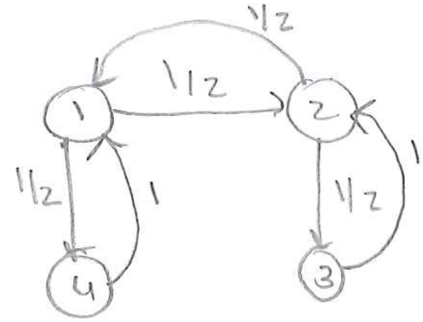


we can see from the diagram that this not an ergodic M.C. The reason is that states 1 and 2 do not communicate with states 3 and 4.

Since it is not ergodic, it follows that it is not regular.

Finally, the chain is not absorbing since there are no absorbing states.

1.(c) The fourth chain diagram is:



From the diagram we can see that this chain IS ergodic.

$1 \to 2$; $2 \to 1$, so states 1 & 2 communicate.

$1 \to 2 \to 3 \Rightarrow 1 \to 3$ and
$3 \to 2 \to 1 \Rightarrow 3 \to 1$ } 1 & 3 communicate

Since 1 & 2 communicate and 1 & 3 communicate $\Rightarrow$ 2 & 3 communicate. Finally, $1 \to 4$; $4 \to 1$ so 1 & 4 communicate, this shows that all states communicate.

The chain is not absorbing since there are no absorbing states. Note that the period of state 1 is 2, since it takes multiples of 2 to get back to it. Since all states communicate, the period of all states is 2. Hence, the chain is periodic with period 2. Therefore, the chain is NOT regular.

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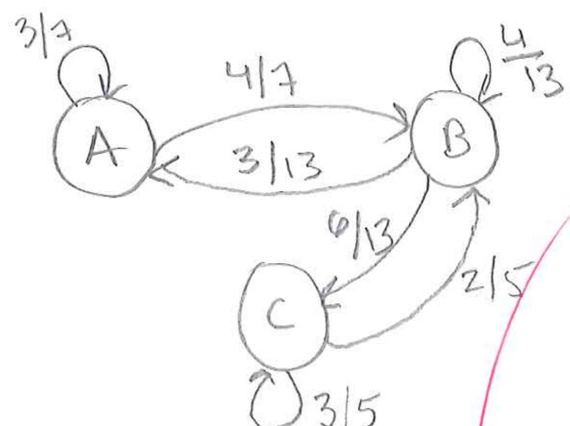
3. Let compartment A be the one with pressure 4, compartment B be the one with pressure 3 and compartment C be the one with pressure 2.
(a)

Then, a diagram for this chain is:



From A we can either stay in A or go to B. Let x be the prob. of staying in A and y of moving to B. Then

$$x : \frac{1}{4} = y : \frac{1}{3} \quad ; \quad x + y = 1$$

$$\frac{x}{y} = \frac{\frac{1}{4}}{\frac{1}{3}} \Rightarrow \frac{x}{y} = \frac{3}{4} \Rightarrow x = \frac{3}{4}y \Rightarrow \frac{3}{4}y + y = 1 \Rightarrow \left(\frac{3}{4}+1\right)y = 1$$

$$\Rightarrow \frac{7}{4}y = 1 \Rightarrow \boxed{y = \frac{4}{7}} \Rightarrow \boxed{x = \frac{3}{7}}$$

A similar calculation goes for all other transitions.

let x be the prob of going from B to B.
y " " " " B to A
z " " " " B to C.
then:

$$x : \frac{1}{3} = y : \frac{1}{4} = z : \frac{1}{2} \; ; \quad x + y + z = 1$$

$$\Rightarrow \frac{x}{y} = \frac{\frac{1}{3}}{\frac{1}{4}} \Rightarrow \frac{x}{y} = \frac{4}{3} \Rightarrow y = \frac{3}{4}x$$

$$\Rightarrow \frac{x}{z} = \frac{\frac{1}{3}}{\frac{1}{2}} \Rightarrow \frac{x}{z} = \frac{2}{3} \Rightarrow z = \frac{3}{2}x$$

$$x + \frac{3}{4}x + \frac{3}{2}x = 1$$

$$\left(1 + \frac{3}{4} + \frac{3}{2}\right)x = 1$$

$$\Rightarrow \left(\frac{4}{4} + \frac{3}{4} + \frac{6}{4}\right)x = 1 \Rightarrow \frac{13}{4}x = 1 \Rightarrow \boxed{x = \frac{4}{13}}$$

So, $y = \frac{3}{4}\frac{4}{13} \Rightarrow \boxed{y = \frac{3}{13}}$ AND $z = \frac{3}{2}\frac{4}{13} \Rightarrow \boxed{\frac{6}{13}}$ z

Finally, let x be the prob of going from C to B AND
y be the prob of going from C to C. then

$$x : \frac{1}{3} = y : \frac{1}{2} \; ; \quad x + y = 1$$

$$\frac{x}{y} = \frac{\frac{1}{3}}{\frac{1}{2}} \Rightarrow \frac{x}{y} = \frac{2}{3} \Rightarrow x = \frac{2}{3}y \Rightarrow \frac{2}{3}y + y = 1 \Rightarrow \left(\frac{2}{3} + 1\right)y = 1$$

$$\Rightarrow \frac{5}{3}y = 1 \Rightarrow \boxed{y = \frac{3}{5}}$$

Hence, $\boxed{x = \frac{2}{5}}$

(b) this chain <u>is ergodic</u>. $A \to B$; $B \to A$, so A & B communicate. $A \to B \to C$; $C \to B \to A$, so A & C communicate that A & B and A & C communicate implies that A & C communicate, so all states communicate

(d) Solving for $wP=w$ and $\sum_{i=1}^{3} w_i = 1$, we get the stable vector w. In the long run, the fraction of time spend in the middle would correspond to $\boxed{w_2 = \frac{52}{133}}$

(c) We want to solve $wP=w$ and $\sum_{i=1}^{3} w_i = 1$.
Note that this chain is regular since it is ergodic (part (b)) and has period 1 (any state can go to itself in one transition).
So we can solve for long-term:

$$[w_1 \; w_2 \; w_3] \begin{bmatrix} 3/7 & 4/7 & 0 \\ 3/13 & 4/13 & 6/13 \\ 0 & 2/5 & 3/5 \end{bmatrix} = [w_1 \; w_2 \; w_3]$$

$$\Rightarrow \begin{cases} \frac{3}{7}w_1 + \frac{3}{13}w_2 = w_1 \\ \frac{4}{7}w_1 + \frac{4}{13}w_2 + \frac{2}{5}w_3 = w_2 \\ \frac{6}{13}w_2 + \frac{3}{5}w_3 = w_3 \end{cases} \Rightarrow \begin{cases} \frac{3}{13}w_2 = \left(1-\frac{3}{7}\right)w_1 = \frac{4}{7}w_1 \Rightarrow w_2 = \frac{52}{21}w_1 \\ \frac{6}{13}w_2 = \left(1-\frac{3}{5}\right)w_3 = \frac{2}{5}w_3 \Rightarrow w_3 = \frac{30}{26}w_2 \end{cases}$$

So, $w_3 = \frac{30}{26}\left[\frac{52}{21}w_1\right] = \frac{1560}{546}w_1$; Now we can solve:

$$w_1 + \frac{52}{21}w_1 + \frac{1560}{546}w_1 = 1 \Rightarrow \left(1 + \frac{52}{21} + \frac{1560}{546}\right)w_1 = 1$$

$$\left(\frac{11{,}466 + 28{,}392 + 32{,}760}{11{,}466}\right)w_1 = 1 \Rightarrow w_1 = \frac{11{,}466}{72{,}618} = \boxed{\frac{3}{19} = w_1}$$

Thus, $w_2 = \frac{52}{21} \cdot \frac{3}{19} = \frac{156}{399} = \boxed{\frac{52}{133} = w_2}$

$$w_3 = \frac{30}{26} \cdot \frac{52}{133} = \frac{1560}{3458} = \boxed{\frac{60}{133} = w_3}$$

6

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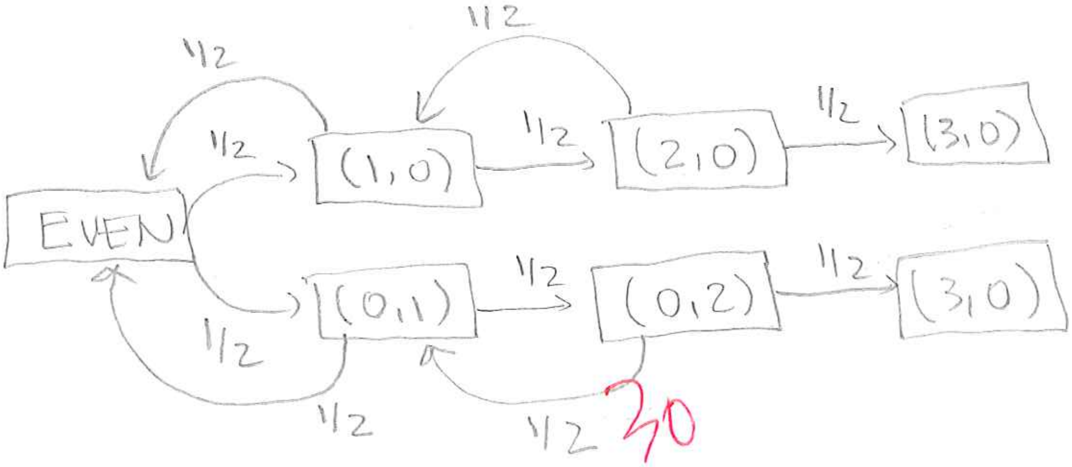
1 (a). The Transition diagram is given by:



where pair (x,y) denotes x denotes the advantage/disadvantage of player 1 and y denotes the advantage/disadvantage of player 2.

(b&c) The transition matrix in canonical form, where states (3,0) and (0,3) are absorbing is:

$$
\begin{array}{c}
 \\ (3,0) \\ (0,3) \\ EVEN \\ (1,0) \\ (0,1) \\ (2,0) \\ (0,2)
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} (3,0) & (0,3) & EVEN & (1,0) & (0,1) & (2,0) & (0,2) \end{array} \\
\left[\begin{array}{cc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 0 & 0
\end{array}\right]
\end{array}
$$

$\underbrace{\qquad\qquad}_{\text{MAtrix } R}$ $\underbrace{\qquad\qquad\qquad\qquad}_{\text{MAtrix } Q}$

(d) the expected number of games that must be played to complete this play off is the expected number of games so that player 1 or player 2 wins.
Our starting state is state $\boxed{\text{EVEN}}$, which is the first non-absorbing state; so just add the numbers on the first row of N: $\boxed{n_{12} + n_{22} + n_{23} + n_{24}}$

(e) the expected number of times that player 1 will be one point away from winning corresponds to state (2,0).
We start at $\boxed{\text{EVEN}}$, so this number is just $\boxed{n_{14}}$

(f) this is just entry a_{21}; that is the probability of ending in absorbing state 1 which is $\boxed{(3,0)}$ (player 1 wins), given that we are in nonabsorbing state 2 which is $\boxed{(1,0)}$, that the first player wins the first game.