M447 - Mathematical Models/Applications 1 - Homework 4

Enrique Areyan
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Chapter 3, Section 3.3

(8) Amy is studying the feeding habits of a certain bird. She observes that the bird always comes the first day she makes food available. After that, however, whenever food is available the pattern of feeding is as follows:

* If the bird feeds one day, then it never feeds the next day.
* If the bird feeds on day $n-1$ and does not feed on day n, then it feeds on day $n+1$ with probability .75 and does not feed on day $n+1$ with probability .25.
* If the bird feeds neither on day $n-1$ nor on day n, then it feeds on day $n+1$ with probability .85 and does not feed on day $n+1$ with probability .15.

(a) Formulate a Markov chain model for this situation.

(b) In the long run, on what fraction of the days does the bird feed?

Solution:

(a) To model this situation, let us consider a Markov Chain whose states are pairs $(M, M-1)$, where M represents the feeding behavior on day M and $M-1$ represents the reading behavior the day before M. These states are sufficient to build a Markov Chain because the behavior of the bird on day $M+1$ depends only on days M and $M-1$. Let F represent a day when the bird feed and N a day in which it did not feed. Then, there are 3 states for the Markov chain, i.e., FN, NF, NN (the state FF is never used), and the transition matrix $\mathbf{P}$ is given by:

$$\mathbf{P} = \begin{array}{c} \\ \text{FN} \\ \text{NF} \\ \text{NN} \end{array} \begin{array}{c} \begin{array}{ccc} \text{FN} & \text{NF} & \text{NN} \end{array} \\ \left\| \begin{array}{ccc} 0 & 1 & 0 \\ .75 & 0 & .25 \\ .85 & 0 & .15 \end{array} \right\| \end{array}$$

(b) To find the long run fraction of the days that the bird feed, let us solve the following linear system, where $\pi = [\pi_0 \quad \pi_1 \quad p_2]$: (note that this is a regular chain)

$$\pi P = \pi, \text{ and } \sum_{i=0}^{2} \pi_i = 1, \text{ from which we get the equations:}$$

$$\begin{array}{lll} 0\pi_0 + 0.75\pi_1 + 0.85\pi_2 = \pi_0 & & \\ \pi_0 = \pi_1 & \Longrightarrow \quad \pi_0 = \pi_1 & \Longrightarrow \\ 0\pi_0 + .25\pi_1 + .15\pi_2 = \pi_2 & \quad .25\pi_1 = .85\pi_2 & \pi_1 = \dfrac{.85}{.25}\pi_2 = \dfrac{17}{5}\pi_2 \end{array}$$

Therefore, we have that $\pi_0 = \pi_1 = \frac{17}{5}\pi_2$. Substituting into the equation $\sum_{i=0}^{2} \pi_i = 1$ we get:

$$\sum_{i=0}^{2} \pi_i = 1 \Longrightarrow \frac{17}{5}\pi_2 + \frac{17}{5}\pi_2 + \pi_2 = 1 \Longrightarrow \left(2 \cdot \frac{17}{5} + 1\right)\pi_2 = 1 \Longrightarrow \pi_2 = \frac{5}{39}$$

This means that $\pi_0 = \pi_1 = \frac{17}{5} \cdot \frac{5}{39} = \frac{17}{39}$. So the fraction of time in each state is

$$\pi = [\pi_0 \quad \pi_1 \quad p_2] = \begin{bmatrix} \frac{17}{39} & \frac{17}{39} & \frac{5}{39} \end{bmatrix}$$

Finally, according to our notation we have that $\pi_0 = FN, \pi_1 = NF$, and $\pi_2 = NN$. Hence, in the long run, the fraction of the days that the bird feed is:

$$FN = \pi_0 = \boxed{\frac{17}{39}}$$

So the birds feeds approximately 43.59% of the time.