M447 - Mathematical Models/Applications 1 - Homework 3

Enrique Areyan

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Chapter 2, Section 2.7

(7) Suppose that for a single-serve queue with exponential arrivals and exponential service distributions, the arrival rate λ suddenly doubles to 2λ, while the service rate μ remains unchanged. Suppose also that the ratio $\frac{\lambda}{\mu}$, which was $\frac{1}{3}$, is now $\frac{2}{3}$. How does the average time spent in the queue change, and how does the average number of units in the queue change?

Solution:

i) For the average time spent in the queue: let W_λ be the waiting time in line before the doubling of λ. Let $W_{2\lambda}$ be the waiting time after the doubling of λ. Then,

$$\begin{aligned}
W_{2\lambda} - W_\lambda &= \left[\frac{2}{3}\left(\frac{1}{\mu - 2\lambda}\right)\right] - \left[\frac{1}{3}\left(\frac{1}{\mu - \lambda}\right)\right] \\
&= \frac{1}{3}\left[\frac{2}{\mu - 2\lambda} - \frac{1}{\mu - \lambda}\right] \\
&= \frac{1}{3}\left[\frac{2(\mu - \lambda) - (\mu - 2\lambda)}{(\mu - 2\lambda)(\mu - \lambda)}\right] \\
&= \frac{1}{3}\left[\frac{\mu}{(\mu - 2\lambda)(\mu - \lambda)}\right] \\
&= \left(\frac{1}{3}\frac{\mu}{\lambda}\frac{\mu}{(\mu - 2\lambda)}\right)\left[\frac{\lambda}{\mu}\frac{1}{\mu - \lambda}\right] && \text{Multiplying by } \frac{\mu}{\lambda}\frac{\lambda}{\mu} = 1 \\
&= \left(\frac{\mu}{\mu - 2\lambda}\right)\left[\frac{1}{3}\frac{1}{\mu - \lambda}\right] && \text{Since } \frac{\mu}{\lambda} = 3 \\
&= \left(\frac{\mu}{\mu - 2\lambda}\right) W_\lambda
\end{aligned}$$

Therefore,

$$W_{2\lambda} - W_\lambda = \left(\frac{\mu}{\mu - 2\lambda}\right) W_\lambda \Longrightarrow W_{2\lambda} = W_\lambda + \left(\frac{\mu}{\mu - 2\lambda}\right) W_\lambda \Longrightarrow W_{2\lambda} = W_\lambda\left(1 + \frac{\mu}{\mu - 2\lambda}\right) \Longrightarrow \boxed{W_{2\lambda} = \left(\frac{2\mu - 2\lambda}{\mu - 2\lambda}\right) W_\lambda}$$

In order for the queue not to explode, we must have $\mu > 2\lambda$. Therefore $\frac{2\mu - 2\lambda}{\mu - 2\lambda} > 1$, so the average time spent in the queue will increase by a factor of $\dfrac{2\mu - 2\lambda}{\mu - 2\lambda}$ relative to W_λ.

ii) For the average number of units in the queue: let L_λ be the length of the line before the doubling of λ. Let $L_{2\lambda}$ be the waiting time after the doubling of λ. Also, let n be the number of people in the system.Then,

$$L_\lambda = \begin{cases} \text{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

We know that $P(L_\lambda = 0) = p_0 + p_1$ and $P(L_\lambda = l) = p_{l+1}$, for $l > 0$ i.e., there are $l + 1$ people in the system so that one is being serve and l are in line. We can compute the expected value of this random variable:

$$
\begin{aligned}
E[L_\lambda] &= \sum_l l \cdot P(L_\lambda = l) && \text{by definition of expected value} \\
&= \sum_{l=0}^{\infty} l \cdot p_{l+1} && \text{by definition of } L_\lambda \\
&= \sum_{l=0}^{\infty} l \cdot \left(\frac{\lambda}{\mu}\right)^{l+1} \left(1 - \frac{\lambda}{\mu}\right) && \text{according to our probabilities } p_n \\
&= 0 + \left(\frac{\lambda}{\mu}\right)^{2} \left(1 - \frac{\lambda}{\mu}\right) \sum_{l=1}^{\infty} l \cdot \left(\frac{\lambda}{\mu}\right)^{l-1} && \text{algebra} \\
&= \left(\frac{\lambda}{\mu}\right)^{2} \left(1 - \frac{\lambda}{\mu}\right) \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{2}} && \text{sum of geometric series}
\end{aligned}
$$

Replacing for the ratio $\frac{\lambda}{\mu} = \frac{1}{3}$ we get:

$$
E[L_\lambda] = \left(\frac{1}{3}\right)^{2} \left(1 - \frac{1}{3}\right) \frac{1}{\left(1 - \frac{1}{3}\right)^{2}} = \frac{1}{9}\frac{2}{3}\frac{1}{\left(\frac{2}{3}\right)^{2}} = \frac{1}{9}\frac{2}{3}\frac{9}{4} = \frac{1}{6}
$$

The same equation holds for $E[L_{2\lambda}]$, we only have to replace the appropriate ratio $\frac{\lambda}{\mu} = \frac{2}{3}$.

$$
E[L_2\lambda] = \left(\frac{2}{3}\right)^{2} \left(1 - \frac{2}{3}\right) \frac{1}{\left(1 - \frac{2}{3}\right)^{2}} = \frac{4}{9}\frac{1}{3}\frac{1}{\left(\frac{1}{3}\right)^{2}} = \frac{4}{9}\frac{1}{3}\frac{9}{1} = \frac{4}{3}
$$

Even tough the arrival rate λ only doubled, the average length of the queue grew by a factor of 8 since:

$$
\boxed{8 \cdot E[L_\lambda] = 8 \cdot \frac{1}{6} = \frac{4}{3} = E[L_{2\lambda}]}
$$

So the new average length is eight times longer than the previous one.