Taxicab Geometry, or When a Circle is a Square

Circle on the Road 2012, Math Festival

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Abstract

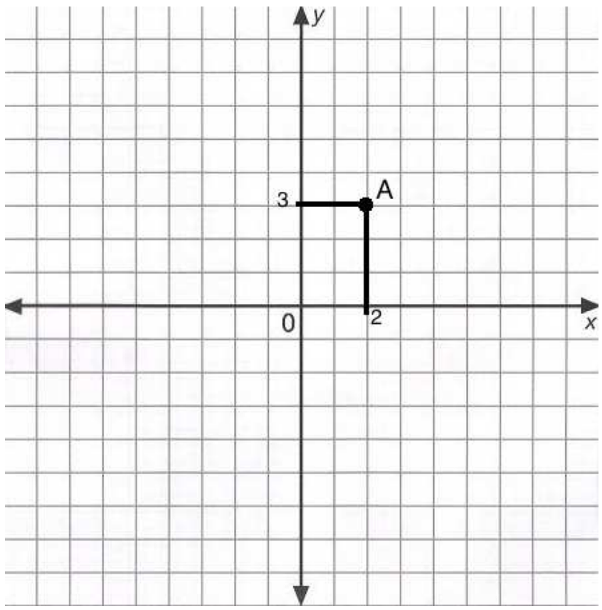
The distance between two points is the length of the shortest path connecting them. In plane Euclidean geometry such a path is along the straight line connecting the two points. In contrast, in a city consisting of a square grid of streets shortest paths between two points are no longer straight lines (as every cab driver knows). We will explore the geometry of this unusual distance and play several related games.

René Descartes (1596-1650) was a French mathematician, philosopher and writer. Among his many accomplishments, he developed a very convenient way to describe positions of points on a plane. This method was very important for future development of mathematics and physics. We will start learning about this invention today.

The city of Descartes is a plane that extends infinitely in all directions:

- The center of the city is marked by point O.
- The horizontal (West-East) line going through O is called the x-axis.
- The vertical (South-North) line going through O is called the y-axis.
- Each house in the city is represented by a point which is the intersection of a vertical and a horizontal line. Each house has an address which consists of two whole numbers written inside of parenthesis.

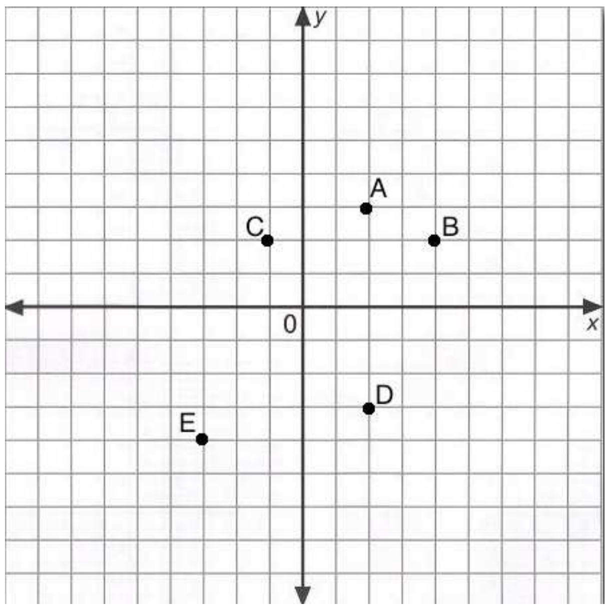
Example. Point A shown below has coordinates $(2, 3)$.



- The first number tells you the distance to the y-axis. The distance is *positive* if you are on the right of the y-axis. The distance is *negative* if you are on the left side of the y-axis.

- The second number tells you the distance to the x-axis. The distance is *positive* if you are above the x-axis. The distance is *negative* if you are below the y-axis.

Problem 1. Find coordinates of several points in the city:



1. Point 0 has address (,); Point A has address (,);
2. Point B has address (,); Point C has address (,);
3. Point D has address (,); Point E has address (,);

Problem 2. Plot several points with given coordinates:

1. Point F with address $(1, 4)$; Point G with address $(4, 1)$;
2. Point H with address $(5, 3)$; Point J with address $(-2, 6)$;
3. Point K with address $(0, -2)$; Point L with address $(3, 0)$.

Game One: Dispatch the Firetruck

A fire starts at some intersection in the city. There are two firetrucks nearby. Decide which firetruck should be dispatched to the fire site depending on the coordinates of the fire and the current positions of the firetrucks. Remember that cars can only drive along the vertical or the horizontal streets in the city. We also assume that both firetrucks travel with the same speed.

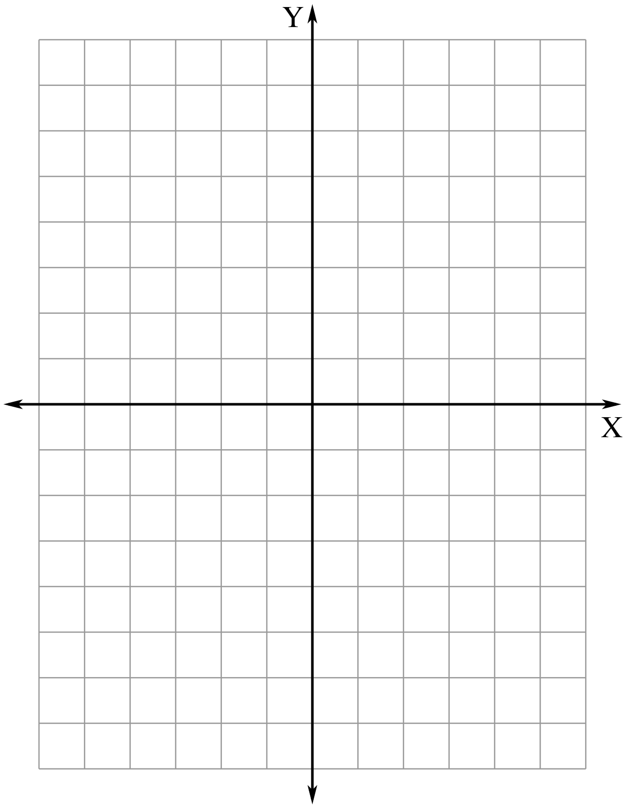
Start by drawing the routes the cars will be taking on the coordinate planes on the next page

1. Fire site: $(0, 0)$;
   First firetruck: $(5, 0)$;
   Second firetruck: $(-4, 0)$.

2. Fire site: $(0, 0)$;
   First firetruck: $(5, 0)$;
   Second firetruck: $(0, 9)$.

3. Fire site $(0, 0)$;
   First firetruck: $(4, 3)$;
   Second firetruck: $(2, 6)$.

Can you give instructions to the fire department dispatcher on how to decide which of the firetrucks should be send to the fire in general?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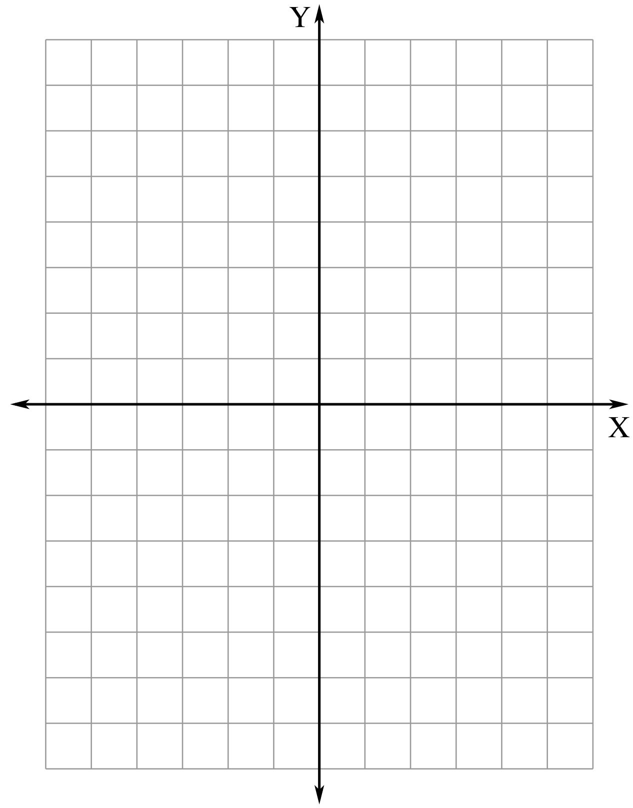
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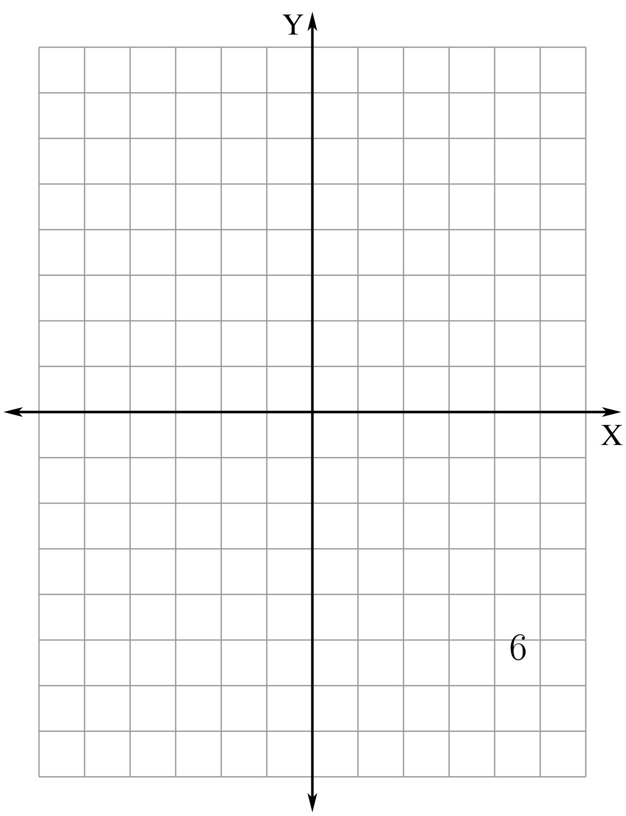
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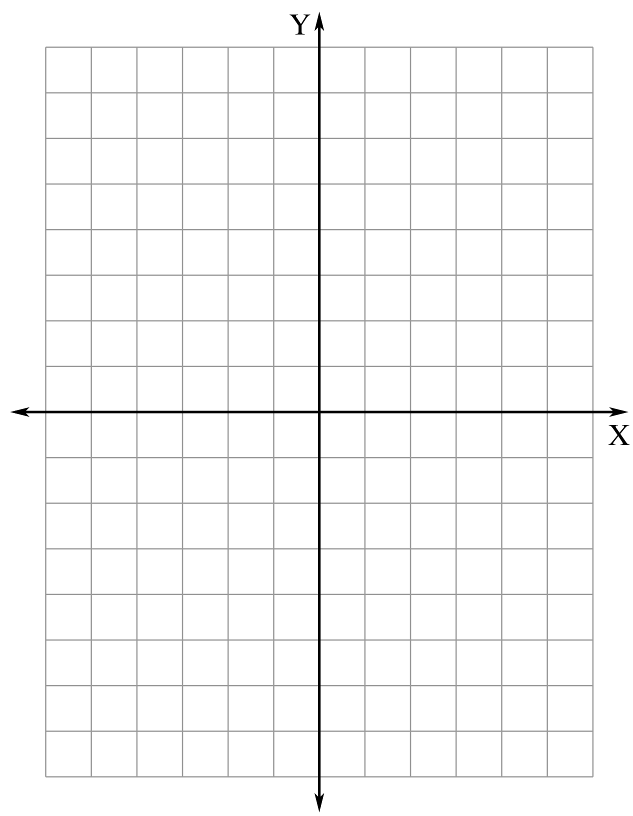
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The Taxicab Distance

Let $A = (a, b)$ and $B = (c, d)$ be two points on the coordinate plane. In the usual plane geometry, the shortest route between these two points is along the straight line connecting them. Let $C = (c, b)$ be a point so that $\triangle ABC$ is a right angle triangle with right angle $\angle C$. The lengths of the two shorter sides of the triangle are

$$\begin{aligned} |AC| &= |a - c| \\ |BC| &= |b - d|, \end{aligned}$$

where $|\ldots|$ denotes the absolute value. The distance between A and B is the length of the third side, which we can find from the Pythagorean theorem:

$$\begin{aligned} |AB|^2 &= |AC|^2 + |BC|^2 \\ |AB|^2 &= (a - c)^2 + (b - d)^2 \\ d(A, B) = |AB| &= \sqrt{(a - c)^2 + (b - d)^2}. \end{aligned}$$

For example, the distance between the points $(0, 3)$ and $(4, 0)$ is $\sqrt{3^2 + 4^2} = 5$.

Imagine that you are only allowed to move along vertical lines and along horizontal lines. (Such is the case in a city which only has streets running in the north-south direction and in the east-west direction). Let's call such a route a *taxi route.*

Problem 1.

1. Draw the shortest possible *taxi route* from point $A = (0, 3)$ to point $B = (4, 0)$.

2. Find the length of this route:

$$d_{\text{taxi}}(A, B) =$$

3. Is there another taxi route that also gives you the shortest possible distance between the two points? Draw several other shortest routes (use different colors).

We will call the distance between points A and B obtained by going along one of the shortest taxi routes the *taxicab distance.*

Problem 2.

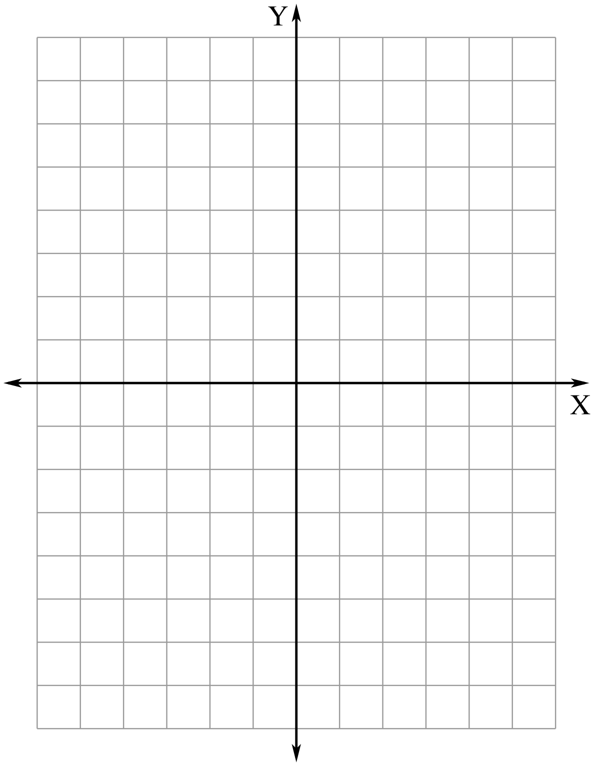
Find the *taxicab distances* between the following points:

1. $(1, 0)$ and $(1, 7)$.

2. $(3, 2)$ and $(5, 2)$.

3. $(4, 3)$ and $(12, 1)$.

4. Can you describe how the taxicab distance is computed in words?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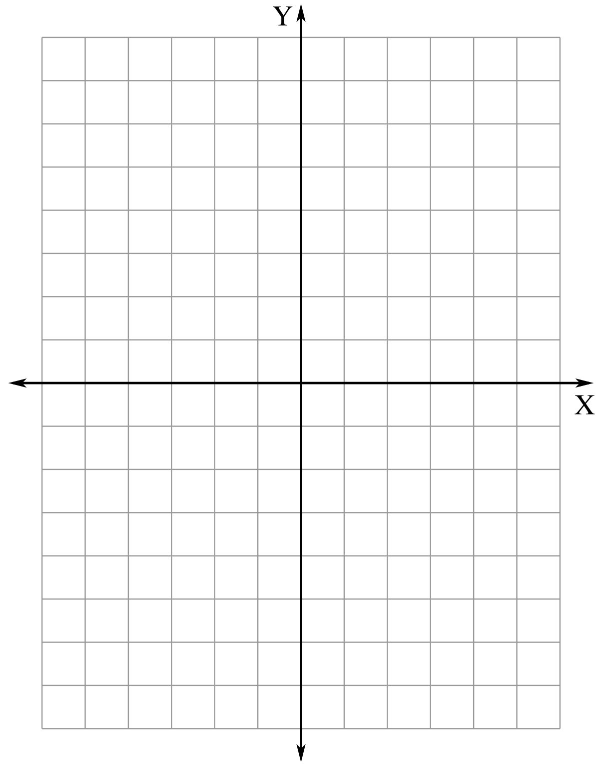
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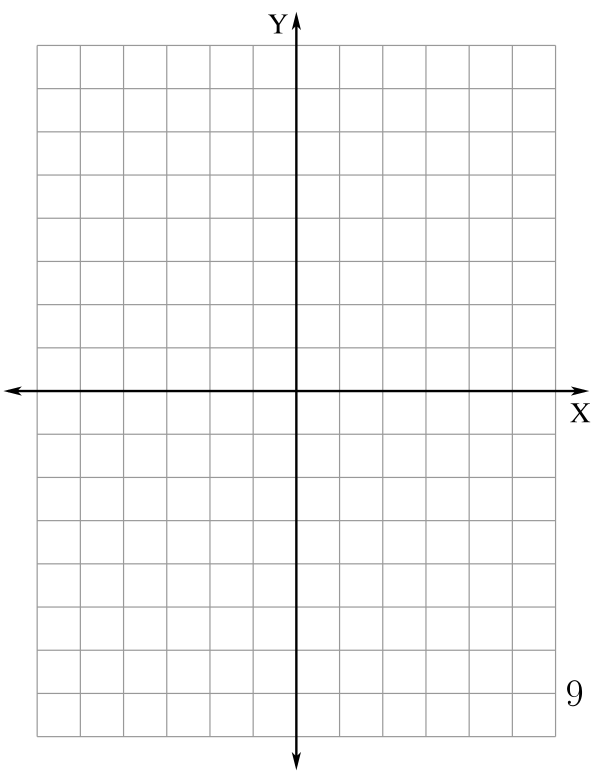
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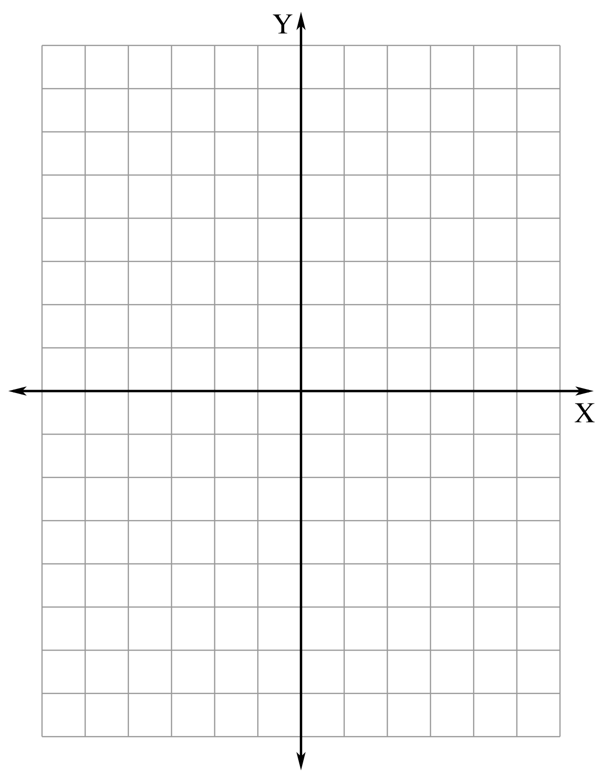
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For points $A = (a, b)$ and $B = (c, d)$, the *taxicab distance* is given by

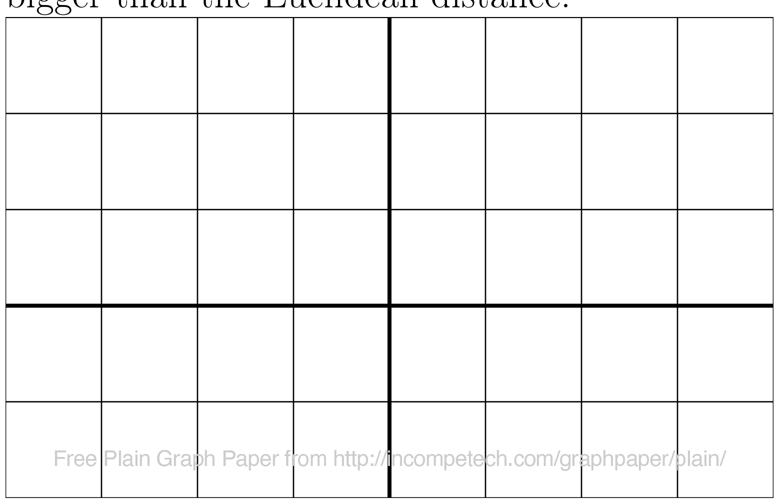
$$d_{\text{taxi}}(A, B) = |a - c| + |b - d|.$$

Here $|a - b|$ denotes the absolute value of the difference between a and c, and $|b - d|$ denotes the absolute value of the difference between b and d.

Problem 3. Let's compare the usual Euclidean distance and the taxicab distance.

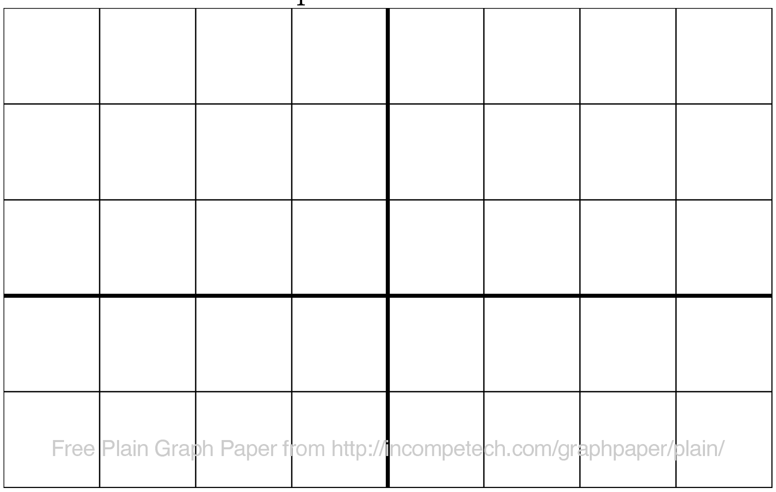
1. Give an example of two points such that the Euclidean (usual) distance and the taxicab distances between them are equal to each other.

2. Give an example of two points such that the taxicab distance is bigger than the Euclidean distance.



3. Can you find a pair of points for which the Euclidean distance is bigger?

Problem 4. Draw the line segment joining points $(2, 0)$ and $(0, 2)$ on the coordinate plane below.



1. What is the point on this segment that is closest to $(0, 0)$ in the usual sense (with respect to the Euclidean distance)?

   - Find the distance from this point to $(0, 0)$:

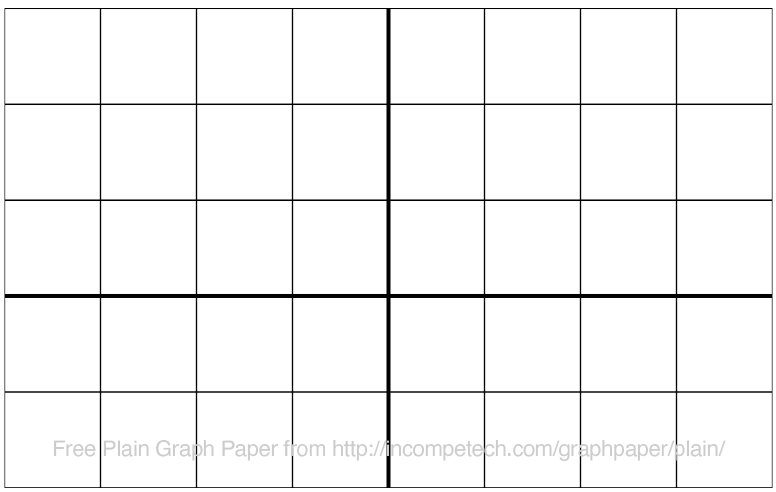
     $$d =$$

   - Find the taxicab distance from this point to $(0, 0)$:

     $$d_{\text{taxi}} =$$

2. Let P be a point with coordinates (x, y) that lies on the line segment connecting $(0, 2)$ and $(2, 0)$. Mark P on the segment and drop therpendiculars from P to the x-axis and the y-axis. Label the lengths of these perpendiculars on the picture. Show that

$$x + y = 2.$$



3. Let P be the same point on the segment connecting $(0, 2)$ and $(2, 0)$. What is the taxicab distance d_{taxi} from $P = (x, y)$ to $(0, 0)$? How does it depend on (x, y)?

4. Make a conclusion about the taxicab distances from various points on the segment to the origin.

Circles and π in Taxicab Geometry

In Euclidean plane geometry, a *circle* can be defined as the set of all points which are at a fixed distance from a given point. The given point is the *center* of the circle. The fixed distance is the *radius* of the circle. *Diameter* is the longest possible distance between two points on the circle and equals twice the radius. *Circumference* is the length of the circle.

The equation of the (Euclidean) circle of radius r centered at point with coordinates (a, b) is

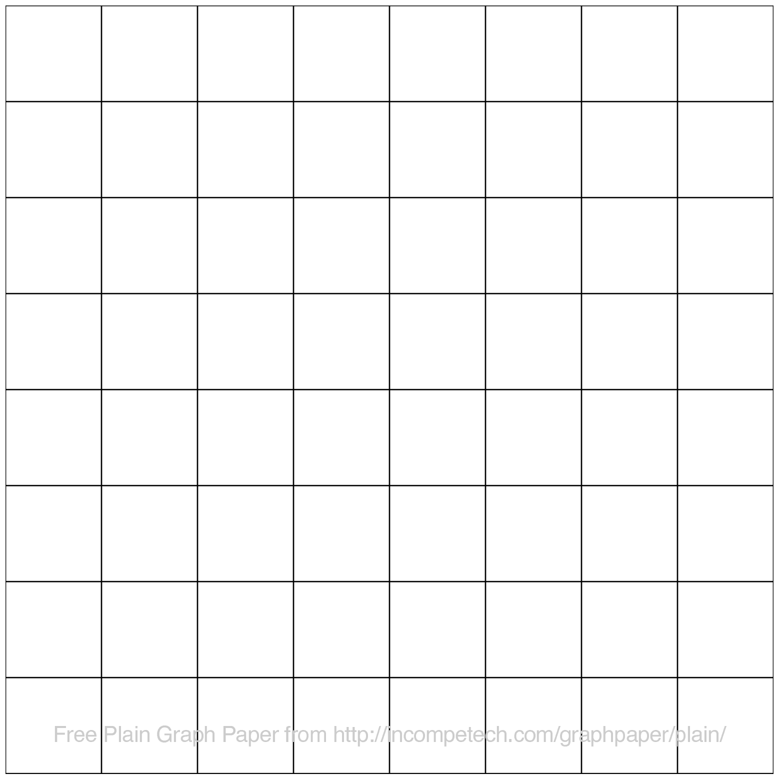
$$(x-a)^2 + (y-b)^2 = r^2.$$

This follows from the formula for the distance.

The same definitions of the circle, radius, diameter and circumference make sense in the taxicab geometry (using the taxicab distance, of course). However, taxicab circles look very different.

Problem 5.

1. The taxicab circle centered at the point $(0, 0)$ of radius 2 is the set of all points for which the taxicab distance to $(0, 0)$ equals to 2. Draw the taxicab circle centered at $(0, 0)$ with radius 2.



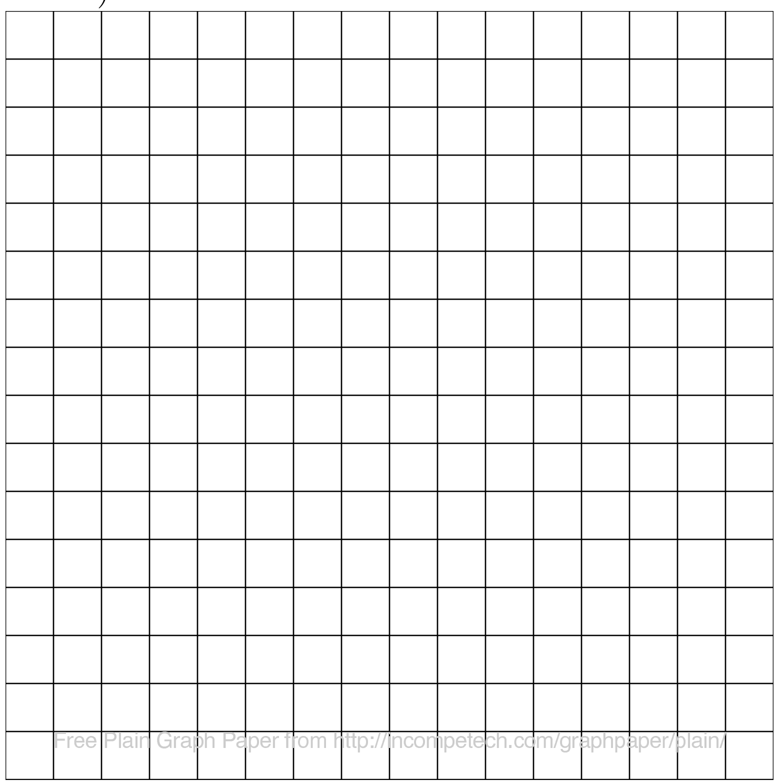
- What is the shape of this taxicab circle?
- Do you think other taxicab circles will have the same shape?

2. Draw the taxicab circle centered at the point $(4, 0)$ of radius 4.

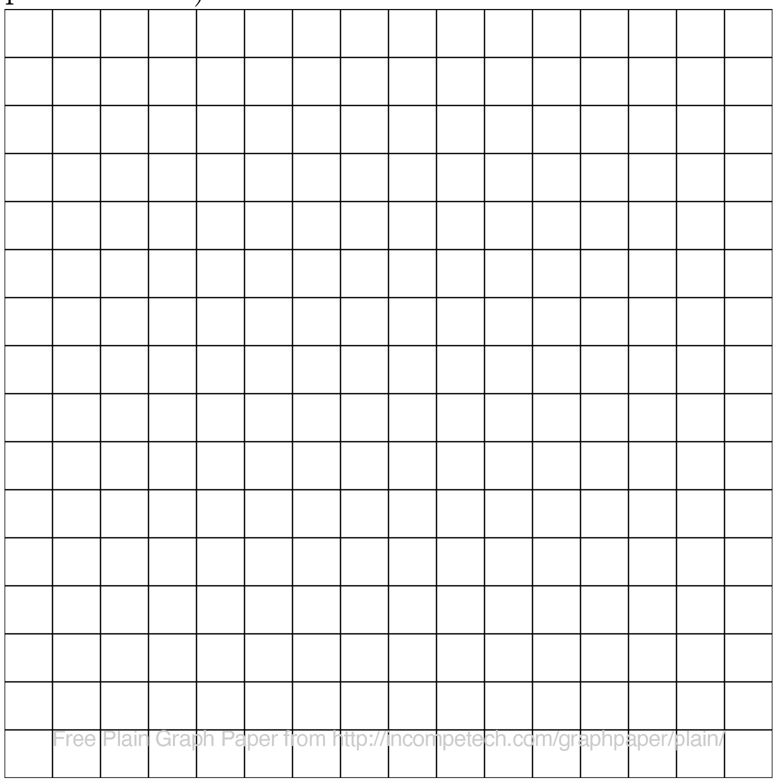
3. Find the coordinates of the points of intersection of these two circles.

Problem 6.

1. Give an example of two taxicab circles which have exactly one common point. (Draw the taxicab circles on the coordinate plane below).



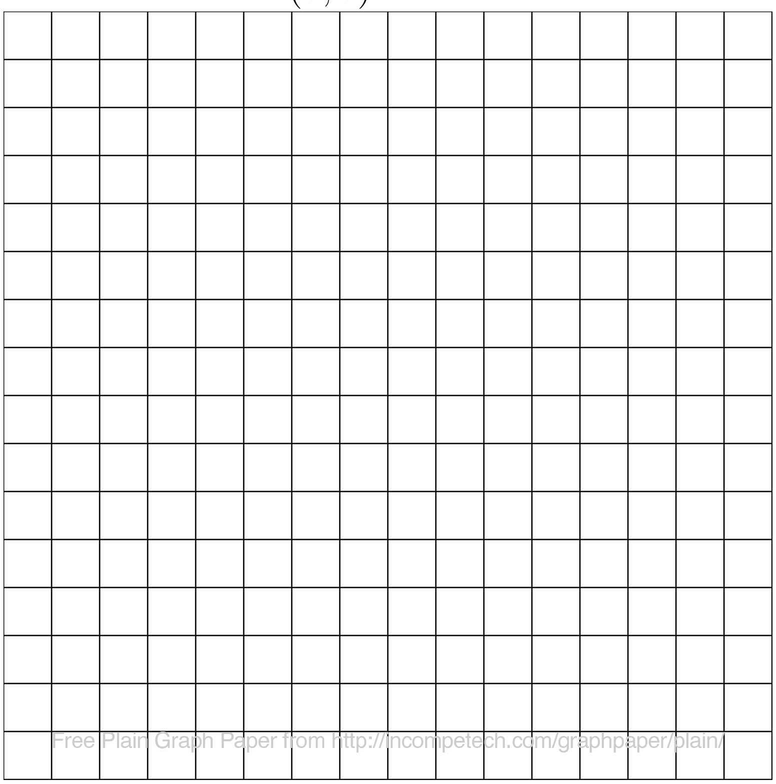
2. Give an example of two distinct taxicab circles which have infinitely many common points. (Draw the taxicab circles on the coordinate plane below).



In Euclidean geometry, π is defined as the ratio of the circumference to the diameter:

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} = \frac{\text{Circumference}}{2 \times \text{Radius}} \simeq 3.14$$

Problem 7. What is the value of π (the ratio of the circumference to the diameter) in taxicab geometry? To find out, consider a taxicab circle centered at $(0, 0)$. Please draw it on the coordinate plane below:



Find the diameter and the taxicab circumference of this circle:

$$\begin{aligned} \text{Taxicab Diameter} &= \\ \text{Taxicab Circumference} &= \end{aligned}$$

Then, compute the ratio

$$\pi_{\text{taxi}} = \frac{\text{Taxicab Circumference}}{\text{Taxicab Diameter}} =$$

Explain why the value of π_{taxi} will be the same for all other taxicab circles.

Compare it with the value of π in the usual Euclidean geometry:

$$\pi_{\text{taxi}} \qquad \pi$$

The equation for the taxicab circle centered at (a, b) and having radius r is

$$|x - a| + |y - b| = r.$$

Game Two: Find the Hidden Treasure

We will play the following game:

Player I (instructor) "hides" the treasure at a certain intersection in the City of Descartes with taxicab distance. (The coordinates of the point are written on a piece of paper and are hidden).

Player II (students) choose a point and asks Player I about the distance from this point to the treasure.

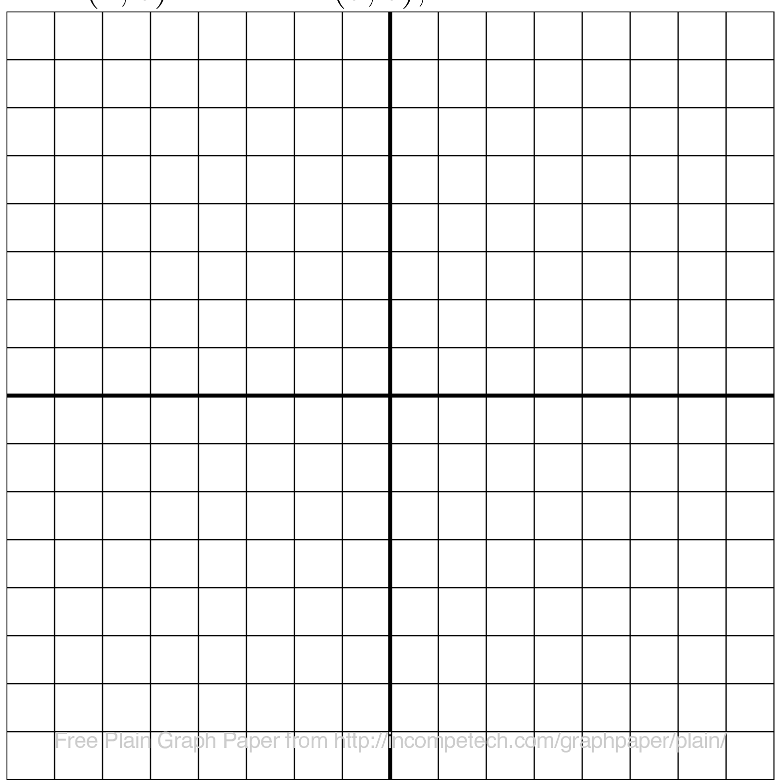
We will play this game several times with the goal of finding out the smallest number of questions Player II needs to find the treasure.

Perpendicular bisectors in Taxicab Geometry

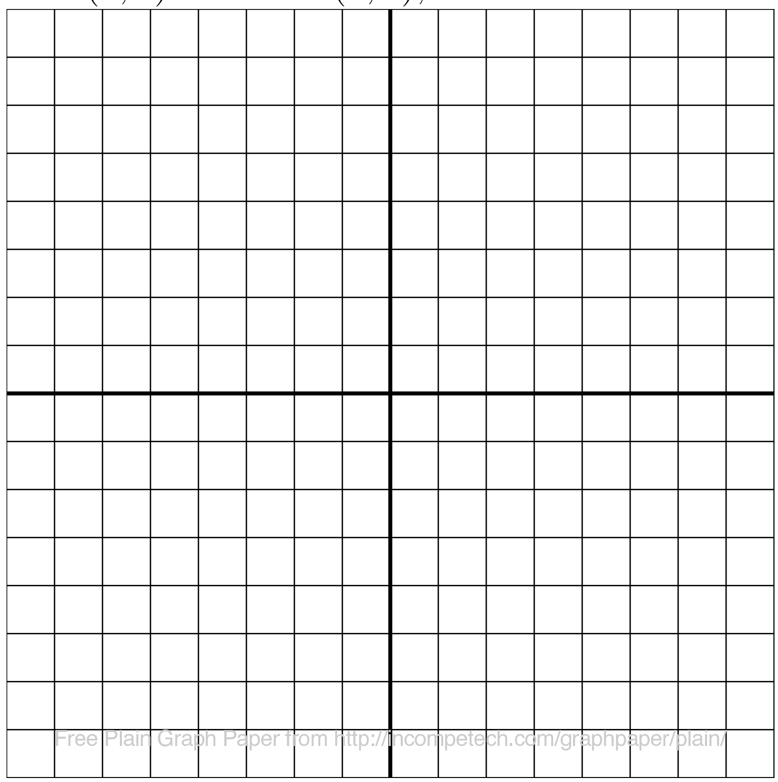
1. Let A and B be points †on the Euclidean plane. Describe the set of all points which are at the same distance from A as they are from B. Draw how this set looks like for several examples of positions of A and B below.

2. Find the set of all points which are at the same taxicab distance from A and from B in the following cases. Draw the set on the coordinate plane:

   (a) $A = (1, 0)$ and $B = (9, 0)$;



(b) $A = (3, 3)$ and $B = (8, 8)$;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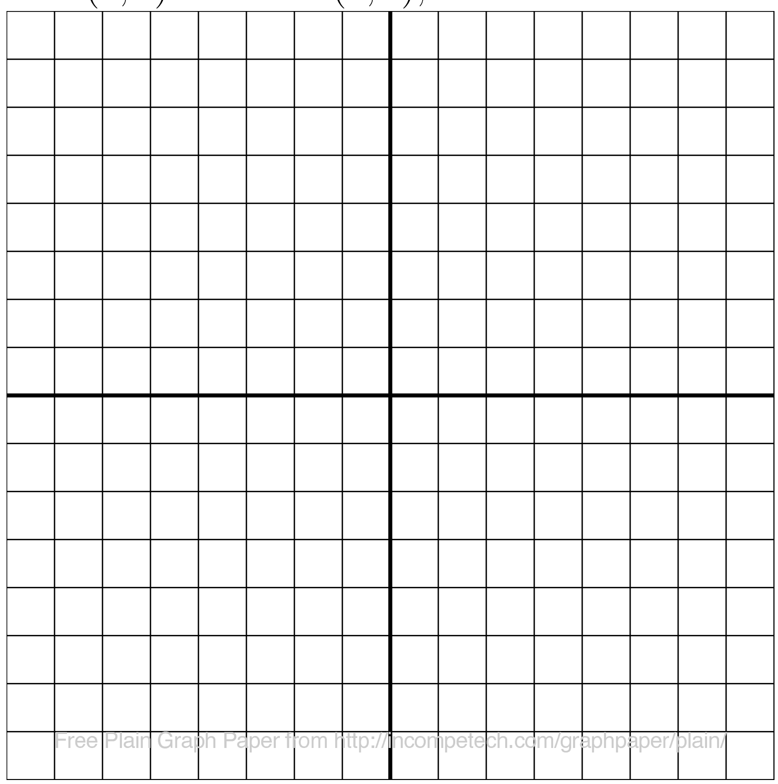
(c) $A = (0, 0)$ and $B = (4, 2)$;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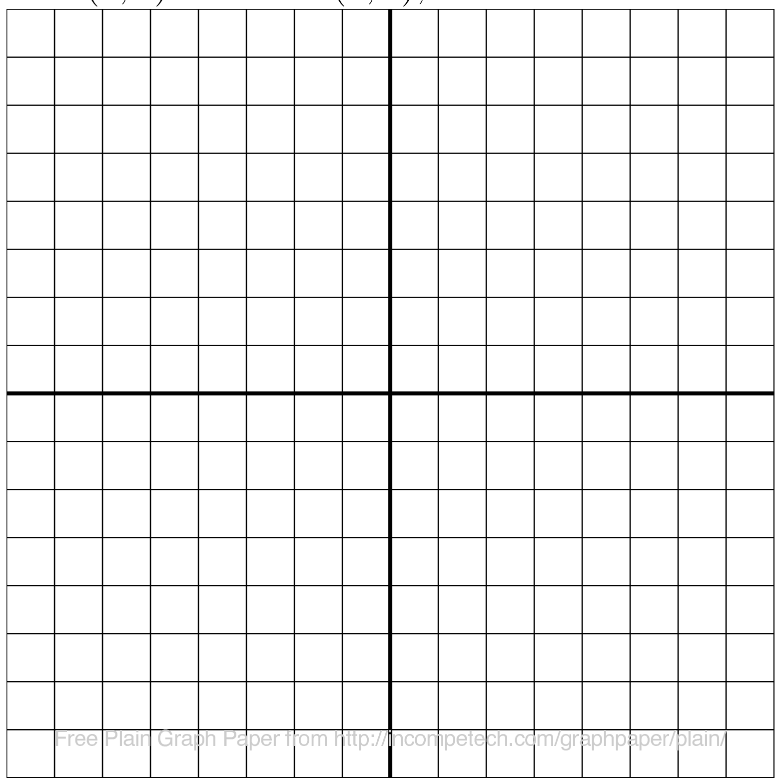
(d) $A = (0, 4)$ and $B = (4, 2)$;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