M436 - Introduction to Geometries - Final Exam

Enrique Areyan
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(Ex. 1) A spherical quadrilateral has edges that are pieces of great circles that are shorter than half a circumference. It tiles the sphere by reflection at its edges. Two of its angles are $72°$. How many of these quadrilaterals are needed to tile the sphere?

Solution: Let us apply a reasoning similar to what we did to determine all triangles that tile the sphere by reflecting across its edges. First, denote the angles of the quadrilateral by

$$\alpha = \frac{2\pi}{A}, \quad \beta = \frac{2\pi}{B}, \quad \gamma = \frac{2\pi}{C} \quad \text{and} \quad \delta = \frac{2\pi}{D}$$

We already know what two of the angles are. So, without loss of generality, suppose that $\gamma = \delta = 72°$. This implies that

$$\gamma = \frac{2\pi}{C} = \delta = \frac{2\pi}{D} = 72° = \frac{2\pi}{5} \Longrightarrow C = D = 5$$

We know that the sum of the angles has to be greater than π. Therefore,

$$\frac{2\pi}{A} + \frac{2\pi}{B} + \frac{2\pi}{5} + \frac{2\pi}{5} > \pi \Longrightarrow 2\pi\left(\frac{1}{A} + \frac{1}{B} + \frac{2}{5}\right) > \pi \Longrightarrow \frac{1}{A} + \frac{1}{B} > \frac{1}{2} - \frac{2}{5} \Longrightarrow \frac{1}{A} + \frac{1}{B} > \frac{1}{10}$$

The numbers A and B have to be greater than 2 for the tiling to close. Changing the inequality to equality we have:

$$\frac{1}{A} + \frac{1}{B} = \frac{1}{10} \Longrightarrow \text{not both } A, B \text{ can be equal to } 20$$

. With these upper bound we can start analyzing cases.

(a) Without loss of generality, suppose $A = 20$. Then, $\frac{1}{20} + \frac{1}{B} > \frac{1}{10} \Longrightarrow \frac{1}{B} > \frac{1}{20} \Longrightarrow B < 20$. In this case we get tilings of the form $(A, B, C, D) = (20, B, 5, 5)$, where $2 < B < 20$.

(b) Suppose A is odd. Then, B has to even. Let $B = 2b$. In this case $\frac{1}{A} + \frac{1}{2b} > \frac{1}{10}$. In this case b has to be greater than 1. So, A is any of $\{3, 5, 7, \ldots 19\}$, and $1 < b < 10$, so we get the tilings $(A, B, 5, 5) = (2k+1, 2b, 5, 5)$, where $1 < k < 10$.

(c) Both A and B even. Then $A = 2a$ and $B = 2b$, the inequality becomes $\frac{1}{A} + \frac{1}{B} > \frac{1}{10} \Longrightarrow \frac{1}{a} + \frac{1}{b} > \frac{1}{5}$. Both a and b are integers greater than 1. Suppose $a = 2$. Then, $\frac{1}{a} + \frac{1}{b} > \frac{1}{5} \Longrightarrow \frac{1}{b} > \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}$, so this case is impossible and does not yield a tiling of quadrilaterals.
Suppose $a = 3$, then $\frac{1}{a} + \frac{1}{b} > \frac{1}{5} \Longrightarrow \frac{1}{b} > \frac{1}{5} - \frac{1}{3} = -\frac{2}{15}$, so this case is impossible and does not yield a tiling of quadrilaterals
Suppose $a = 4$, then $\frac{1}{a} + \frac{1}{b} > \frac{1}{5} \Longrightarrow \frac{1}{b} > \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$, so this case is impossible and does not yield a tiling of quadrilaterals
Suppose $a = 5$, then $\frac{1}{a} + \frac{1}{b} > \frac{1}{5} \Longrightarrow \frac{1}{b} > \frac{1}{5} - \frac{1}{5} = 0 \Longrightarrow b > 0$. This case yields tiling of $(A, B, 5, 5) = (10, 2b, 5, 5)$., where $1 < b < 10$.

The case of a quadrilateral tiling reduces to the case of spherical triangles, since if we take a spherical quadrilateral and divided with a great circles across two of its edges, we would obtain two spherical triangles tiling the sphere, which we already analyzed in class. Hence, the number of quadrilateral tiling is an integer function of the number of triangles from which it is composed. In fact, each quadrilateral is composed of 4 spherical triangles by subdividing the quadrilateral by pieces of 2 great circles through appropriate vertices. By Von Dyck's theorem the number of triangles needed to tile the sphere is $4/(d-1)$ where $d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, so, the number of quadrilaterals needed to tile the sphere is 4 more than this number, i.e., $1/(d-1)$, which in our case $d = \frac{1}{A} + \frac{1}{B} + \frac{2}{5}$, for appropriate choices of A and B as shown above.

Each exercise is on a new page

(Ex. 2) Here are the rules for a lottery:

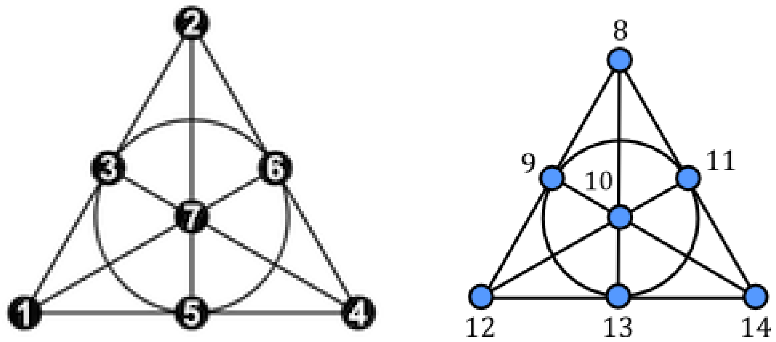
- You can purchase any number of tickets.
- Each ticket has the numbers 1-14 printed on it.
- To validate a ticket, you mark three different numbers from 1 to 14.
- After all tickets are validated, three different numbers from 1 to 14 are chosen randomly.
- Any ticket that has at least two numbers right will win a price in the lottery.

You will show that by purchasing 14 tickets and selecting the numbers wisely, you can ensure to have at least one winning ticket. Draw two Fano planes, and label the vertices in the first plane 1 through 7, and in the second plane 8 through 14. For each of the 14 lines in both Fano planes, purchase a ticket and mark on this ticket the numbers of the vertices that are on that line.

Explain why you must have at least one winning ticket.

Finite Math Note: The probability to win with a single ticket is about 9.3%. When you fill out 14 tickets randomly, the probability that at least one of your tickets wins becomes 74.67%. So what do we learn? Finite Geometry beats Finite Math!

Solution: Let us draw two Fano planes, and label the vertices in the first plane 1 through 7, and in the second plane 8 through 14. The planes are show below with the first one having black vertices and the second one blue vertices.



Consider lines in these planes as tickets bought, where I will represent a ticket by a triplet $(m, n, l), m, n, l \in \{1, 2, 3, \ldots, 14\}$ corresponding to the numbers marked in the ticket. From the planes we can see that we have these tickets:

$$\begin{aligned} \text{From the first plane we get:} \quad & \{(1,2,3),(1,5,4),(1,7,6),(3,7,4),(2,6,4),(3,5,6),(2,7,5)\} \\ \text{From the second plane we get:} \quad & \{(8,9,12),(8,11,14),(8,10,13),(12,13,14),(9,10,14),(10,11,12),(9,11,13)\} \end{aligned}$$

Claim: We have at least one winning ticket.
Proof: Partition the numbers of the lottery into to sets high (H) and (L), where $L = \{1, 2, 3, 4, 5, 6, 7\}$ and $H = \{8, 9, 10, 11, 12, 13, 14\}$. Clearly, the winning ticket must contain at least two numbers from either H or L. But no matter which two numbers are chosen from either H or L, we will have a ticket with these two numbers. This is the case because any two numbers in one of the above Fano planes are connected by a line. Another way to view this is that any number in a Fano plane is connected with a line to any other number. Since any ticket that has at least two numbers right will win a price in the lottery, we are guaranteed to win a price.

Each exercise is on a new page

(Ex. 3) The figure shows the projective plane over the finite field $\mathbb{F}_5$ with 5 elements, with the orderly arranged square depicting the affine plane $z = 1$. The six marked points are the points of the conic $\{(x : y : z) : x^2 + y^2 = z^2\}$, and the line is the tangent line of the conic at the point $(2 : 1 : 0)$. Note that parallel lines are in fact only a single line passing through the same point at infinity.

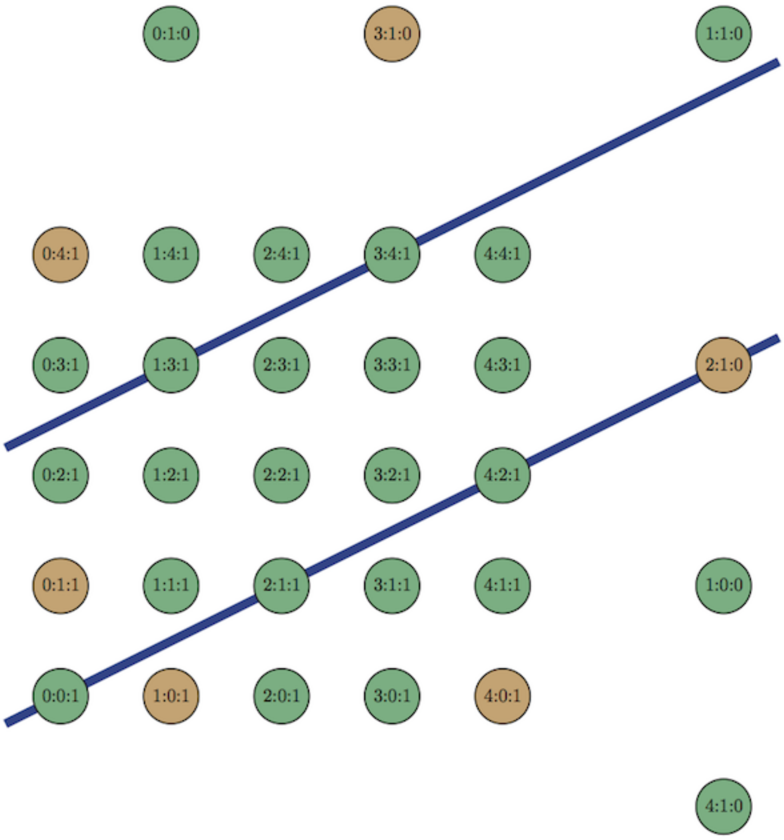


Figure 1 A conic in $\mathbf{F}_5\mathbf{P}^2$

Draw the tangent lines through the other five points of the conic. Explain in one case how you found that tangent line. You will observe that no three different tangent lines can meet at a single point. Show that this is the case for a general, non-degenerate conic in any projective plane over an arbitrary field $\mathbb{F}$.

Hint: Look at the dual projective plane.

Solution: The conic $x^2 + y^2 = z^2 \iff x^2 + y^2 - z^2 = 0$ is given by the symmetric matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ because:

$$0 = x^T C x = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x \\ y \\ -z \end{pmatrix} = x^2 + y^2 - z^2$$

Note that this is a an invertible symmetric matrix and hence the conic is non-degenerate. Therefore, to find the tangent line to the conic a point p in homogenous coordinates we just take Ap. So, for all six points we have:

(i) $(2 : 1 : 0) \implies A \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, hence the line is $2x + y = 0$. Some points on this line are:

$$\{(2 : 1 : 0), (4 : 2 : 1), (2 : 1 : 1), (0 : 0 : 1), (3 : 4 : 1), (1 : 3 : 1)\}$$

(ii) $(3:1:0) \implies A\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}3\\1\\0\end{pmatrix}$, hence the line is $3x + y = 0$. Some points on this line are:

$$\{(3:1:0),(2:4:1),(4:3:1),(1:2:1),(3:1:1),(0:0:1)\}$$

(iii) $(0:4:1) \implies A\begin{pmatrix}0\\4\\1\end{pmatrix} = \begin{pmatrix}0\\4\\-1\end{pmatrix}$, hence the line is $4y - z = 0$. Some points on this line are:

$$\{(0:4:1),(3:4:1),(4:4:1),(2:4:1),(1:4:1),(1:0:0)\}$$

(iv) $(0:1:1) \implies A\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$, hence the line is $y - z = 0$. Some points on this line are:

$$\{(0:1:1),(1:1:1),(2:1:1),(3:1:1),(4:1:1),(1:0:0)\}$$

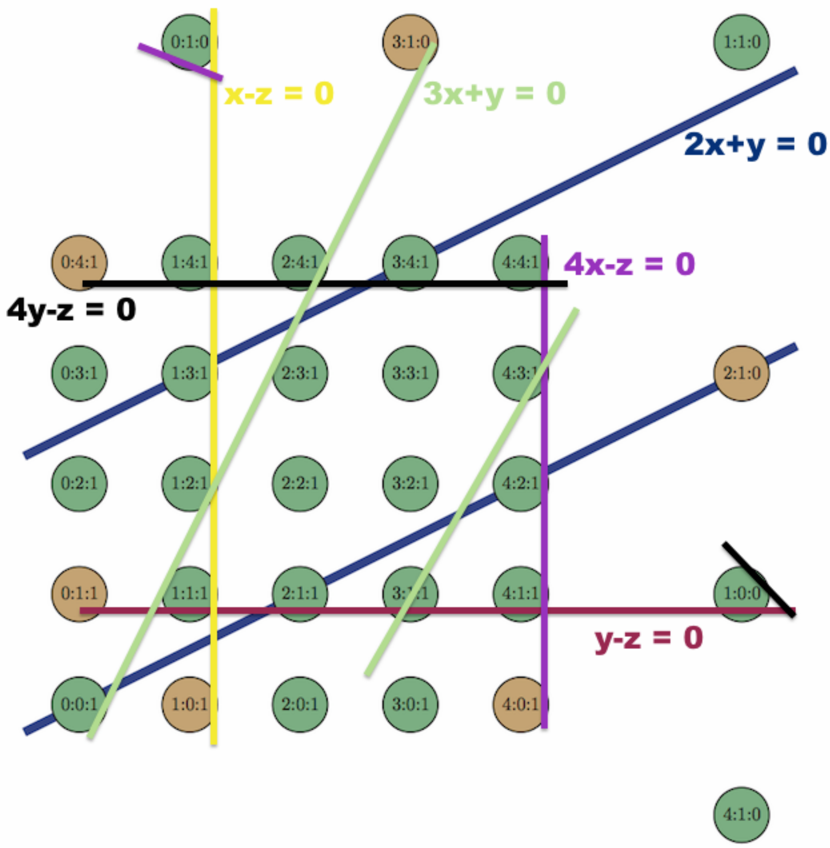
(v) $(1:0:1) \implies A\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, hence the line is $x - z = 0$. Some points on this line are:

$$\{(1:0:1),(1:1:1),(1:2:1),(1:3:1),(1:4:1),(0:1:0)\}$$

(vi) $(4:0:1) \implies A\begin{pmatrix}4\\0\\1\end{pmatrix} = \begin{pmatrix}4\\0\\-1\end{pmatrix}$, hence the line is $4x - z = 0$. Some points on this line are:

$$\{(4:0:1),(4:1:1),(4:2:1),(4:3:1),(4:4:1),(0:1:0)\}$$

The next picture shows the tangent lines through all six points:



Finally, note that no three different tangent lines can meet at a single point. This is the case in general for a non-degenerate conic in any projective plane over an arbitrary field $\mathbb{F}$. For suppose that we had a non-degenerate conic A with three different tangent lines meeting at a single point. Then, in the dual projective plane lines become points and points become lines and hence, we would have a conic A^{-1} with three distinct collinear points. By proposition 1 of the diary, page 84, our conic A^{-1} is then the union of two lines and thus degenerate. But this is a contradiction to the fact that A and A^{-1} are non-degenerate. Therefore, no three different tangent lines can meet at a single point.

Each exercise is on a new page

(Ex. 4) Show that the Mobius transformation

$$f(z) = \frac{-5z+8}{z-7}$$

moves points that lie on the geodesic that foots on the real axis at -2 and 4 by the hyperbolic distance $cosh^{-1}(5/3)$

Solution: Here is the game plan: (1) I will show that the given Mobius transformation f maps the geodesic that foots on the real axis at -2 and 4 back to itself. (2) I will show that, for a particular point on said geodesic f moves it by the hyperbolic distance $cosh^{-1}(5/3)$.

Together (1) and (2) imply that all points on the geodesic are moved by the hyperbolic distance $cosh^{-1}(5/3)$. This is true because f is an invertible Mobius transformation (indeed: $det[\begin{pmatrix} -5 & 8 \\ 1 & -7 \end{pmatrix}] = 27 \neq 0$), and we know that Mobius transformations map circles to circles. Therefore, if (1) holds, f is a rotation of the geodesic which means that all points are moved the same distance. It will suffice only to show that one point moves by the hyperbolic distance $cosh^{-1}(5/3)$ to show that all points move by the same distance.

So let us get started. Clearly the geodesic that foots on the real axis at -2 and 4 is given in cartesian coordinates as $(x-1)^2 + y^2 = 3^2$ where $y > 0$. In other words, the circle that is perpendicular to the real axis and that contains the points $(-2, 0)$ and $(4, 0)$ is the circle centered at $(1, 0)$ with radius 3.

For (1) let us consider an arbitrary point on the geodesic. This is of the form $(x, \sqrt{9-(x-1)^2})$, for $2 < x < 4$. This points gets mapped by f to:

$$\begin{aligned} f((x, \sqrt{9-(x-1)^2})) = f(x + \sqrt{9-(x-1)^2}i) &= \frac{-5(x+\sqrt{9-(x-1)^2}i)+8}{(x+\sqrt{9-(x-1)^2}i)-7} \\ &= \frac{(8-5x)+5\sqrt{9-(x-1)^2}i}{(x-7)+\sqrt{9-(x-1)^2}i} \cdot \frac{(x-7)-\sqrt{9-(x-1)^2}i}{(x-7)-\sqrt{9-(x-1)^2}i} \\ &= \frac{(32-11x)-9\sqrt{-(x-4)(x+2)}i}{4x-19} \end{aligned}$$

To show that this point gets mapped back into the circle of radius 3 centered at $(1, 0)$, it suffices to show that the distance between this point and $(1, 0)$ is 3. We proceed as follow:

$$\begin{aligned} \left| f((x, \sqrt{9-(x-1)^2})) - 1 \right| = \left| \frac{(32-11x)-9\sqrt{-(x-4)(x+2)}i}{4x-19} - 1 \right| &= \left| \frac{(51-15x)-9\sqrt{-(x-4)(x+2)}i}{4x-19} \right| \\ &= \sqrt{\frac{(51-15x)^2 + (-9\sqrt{-(x-4)(x+2)})^2}{(4x-19)^2}} \\ &= \sqrt{\frac{225x^2-1530x+2601-81(x-4)(x+2)}{16x^2-152x+361}} \\ &= \sqrt{\frac{225x^2-1530x+2601-81x^2+162x+648}{16x^2-152x+361}} \\ &= \sqrt{\frac{144x^2-1368x+3249}{16x^2-152x+361}} \\ &= \sqrt{\frac{9(16x^2-152x+361)}{16x^2-152x+361}} \\ &= \sqrt{9} = 3 \end{aligned}$$

This shows (1), i.e., that the image of an arbitrary point on the geodesic is again a point on the geodesic since it is back on the circle of radius 3 centered at $(1,0)$.

For (2) we use the cartesian equation of the geodesic (what is the same, the equation of the circle) and immediately see that the point $(1,3)$ is a point that lies on it since it satisfies $(1-1)^2+3^2=3^2$ and $3>0$. Now, this point gets mapped to the point $(-7/5, 9/5)$ by f since:

$$\begin{aligned}
f((1,3)) = f(1+3i) &= \frac{-5(1+3i)+8}{(1+3i)-7} \\
&= \frac{-5-15i+8}{1-7+3i} \\
&= \frac{3(1-5i)}{3(-2+i)} \\
&= \frac{1-5i}{-2+i}\cdot\frac{-2-i}{-2-i} \\
&= \frac{-2-i+10i+5i^2}{2^2-i^2} \\
&= \frac{-7+9i}{5} \\
&= (-7/5, 9/5)
\end{aligned}$$

The hyperbolic distance between the point $z=(1,3)$ and its image $f(z)=(-7/5,9/5)$ is given by:

$$\begin{aligned}
d_H((1,3),(-7/5,9/5)) &= cosh^{-1}\left[1+\frac{\left(-\frac{7}{5}-1\right)^2+\left(\frac{9}{5}-3\right)^2}{2\cdot 3\cdot\frac{9}{5}}\right] \\
&= cosh^{-1}\left[1+\frac{\left(\frac{12}{5}\right)^2+\left(\frac{6}{5}\right)^2}{\frac{54}{5}}\right] \\
&= cosh^{-1}\left[1+\frac{\frac{144+36}{25}}{\frac{54}{5}}\right] \\
&= cosh^{-1}\left[1+\frac{180\cdot 5}{25\cdot 54}\right] \\
&= cosh^{-1}\left[1+\frac{180}{270}\right] \\
&= cosh^{-1}\left[\frac{450}{270}\right] \\
&= cosh^{-1}\left[\frac{5}{3}\right]
\end{aligned}$$

This shows (2), which completes the proof.

(Ex. 5) Let

$$SL_2(\mathbb{F}_3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_3, ad - bc \equiv 1 \quad (mod \quad 3) \right\}$$

be the set of 2×2 matrices with entries in the field of 3 elements and determinant 1 (modulo 3). For each such matrix, we associate the Mobius transformation

$$f(z) = \frac{az + b}{cz + d}$$

You obtain 12 different Mobius transformations this way; they form a group under composition, using modular arithmetic for the basic operations involving the coefficients. They can also be considered as maps of the set $\mathbb{F}_3 \cup \{\infty\}$ with the provision that any attempted division by 0 has the result ∞, and $f(\infty) = a/c$. Draw the Cayley graph of this group with respect to the set

$$f(z) = f^{-1}(z) = \frac{2}{z}, \qquad g(z) = \frac{1}{2z+1}, \qquad and \qquad g^{-1}(z) = \frac{2z+1}{2z}$$

Solution: The following table contains the 8 elements generated by composing elements of this group. Elements 9,10,11 and 12 are f, g, g^{-1} and the identity.

$$fg = f\left(\frac{1}{2z+1}\right) = \frac{2}{\frac{1}{2z+1}} = z + 2$$

$$fg^{-1} = f(g^{-1}(z)) = f\left(\frac{2z+1}{2z}\right) = \frac{2}{\left(\frac{2z+1}{2z}\right)} = \frac{z}{2z+1}$$

$$fgf = f\left(\frac{z}{z+1}\right) = \frac{2}{\frac{z}{z+1}} = \frac{2z+2}{z}$$

$$ggf = g\left(\frac{z}{z+1}\right) = \frac{1}{2\left(\frac{z}{z+1}\right)+1} = z + 1$$

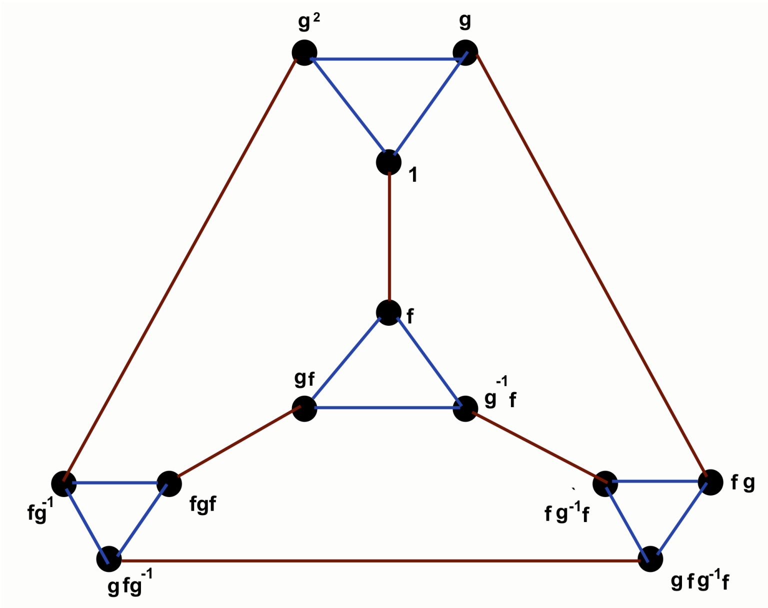
$$gf = g\left(\frac{2}{z}\right) = \frac{1}{2\left(\frac{2}{z}\right)+1} = \frac{z}{z+1}$$

$$gfg = g(z+2) = \frac{1}{2(z+2)+1} = \frac{1}{2z+2}$$

$$gfg^{-1} = g\left(\frac{z}{2z+1}\right) = \frac{1}{2\left(\frac{z}{2z+1}\right)+1} = \frac{2z+1}{z+1}$$

$$g^{-1}fg^{-1} = g^{-1}\left(\frac{z}{2z+1}\right) = \frac{2\left(\frac{z}{2z+1}\right)+1}{2\left(\frac{z}{2z+1}\right)} = \frac{z+1}{2z}$$

The following is the Cayley Graph:



Note that this group is isomorphic to the Alternating group A4 the group of even permutations on four elements. I will not provide a rigorous proof of this fact here, but from the Cayley graph it is pretty clear that this is the case. (A formal proof would require one to construct an isomorphism between this group an A4).