M436 - Introduction to Geometries - Homework 9

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All problems below will use the stereographic projection in the form:

$$\sigma : S^n \mapsto \mathbb{R}^n$$

$$(p1, \ldots, p_{n+1}) \mapsto \frac{1}{1-p_{n+1}}(p_1, \ldots, p_n)$$

Note that there are other conventions.

(Ex. 1) Define a map $i : C \mapsto C$ by first using the inverse stereographic projection to obtain a point in S^2, then applying a reflection at the plane $z = 0$, and finally applying the stereographic projection to get a point in $\mathbb{R}^2 = C$. Show that this map is given by $z \mapsto 1/\bar{z}$

Solution: Note that in general $\sigma^{-1} : \mathbb{R}^n \mapsto S^n$ (the inverse stereographic projection) is given by:

$\sigma^{-1}(q) = \dfrac{2}{|q|^2+1}q + \dfrac{|q|^2-1}{|q|^2+1}e_{n+1}$, where e_{n+1} is the corresponding vector for the standard basis and $q = (q_1, \ldots, q_n, 0)$

In this case we are considering $\sigma : S^2 \mapsto \mathbb{R}^2$. Hence, for $q = (a, b)$ a point in $C = \mathbb{R}^2$, the inverse stereographic projection $\sigma^{-1} : \mathbb{R}^2 \mapsto S^2$ can be written as:

$$\sigma^{-1}((a,b)) = \frac{2}{|(a,b)|^2+1}(a,b,0) + \frac{|(a,b)|^2-1}{|(a,b)|^2+1}(0,0,1) = \frac{1}{a^2+b^2+1}(2a, 2b, a^2+b^2-1)$$

Let r be the reflection at the plane $z = 0$. Then r amounts to changing the sign of the third coordinate:

$$r[\sigma^{-1}((a,b))] = \frac{1}{a^2+b^2+1}(2a, 2b, -(a^2+b^2-1)) = \frac{1}{a^2+b^2+1}(2a, 2b, 1-a^2-b^2)$$

Finally, use σ to project back onto C:

$$\begin{aligned}
\sigma\{r[\sigma^{-1}((a,b))]\} &= \frac{1}{a^2+b^2+1}\left[\frac{1}{1-\dfrac{1-a^2-b^2}{a^2+b^2+1}}(2a,2b)\right] \\
&= \frac{1}{a^2+b^2+1}\left[\frac{1}{\dfrac{2a^2+2b^2}{a^2+b^2+1}}(2a,2b)\right] \\
&= \frac{1}{a^2+b^2+1}\left[\frac{a^2+b^2+1}{2a^2+2b^2}(2a,2b)\right] \\
&= \frac{2}{2(a^2+b^2)}(a,b) \\
&= \frac{1}{a^2+b^2}(a,b)
\end{aligned}$$

Which shows that the map $i : C \mapsto C$ given by $i = \sigma\{r[\sigma^{-1}((a,b))]\}$ indeed is the map $z \mapsto 1/\bar{z}$ since

$$z = (a,b) \mapsto \frac{1}{\overline{(a,b)}} = \frac{1}{a-bi} \cdot \frac{a+bi}{a+bi} = \frac{a+bi}{a^2+b^2} = \frac{1}{a^2+b^2}(a,b)$$

(Ex. 2) Let $u \in S^n \subset \mathbb{R}^{n+1}$ be a unit vector, and d a real number. Recall that the set $\{x : u \cdot x = d\}$ describes a hyperplane perpendicular to n at distance d from the origin. Show that the intersection of this hyperplane with S^n is mapped by the stereographic projection to a sphere in $\mathbb{R}^{n-1}$ with center at $\frac{1}{d-u_{n+1}}(u_1, \ldots, u_n)$ and radius $\frac{\sqrt{1-d^2}}{|u_{n+1}-d|}$.

Solution: Let $H_d(u) = \{p \in \mathbb{R}^{n+1} : p \cdot u = d\}$. For a given unit vector u, the set $H_d(u)$ is the plane perpendicular to u at distance d from the origin. Let σ be the stereographic projection as defined in the beginning of this document. Again, note that σ^{-1} is given by:

$$\sigma^{-1}(q) = \frac{2}{|q|^2+1}q + \frac{|q|^2-1}{|q|^2+1}e_{n+1}, \qquad \text{where } e_{n+1} \text{ is the corresponding vector for the standard basis and } q = (q_1, \ldots, q_n, 0)$$

We want to show that for u a unit vector in $\mathbb{R}^{n+1}$ and $d \in \mathbb{R}$ the set $H_d(u) \cap S^n$ is mapped by σ to a circle with the proposed center and radius. To begin, let $p \in H_d(u) \cap S^n$ so that p is a point representing a "circle" in the sphere S^n. Consider its image under σ: $q = \sigma(p)$. By definition, we must have that $\sigma^{-1}(q) \cdot u = d$. Then,

$$\begin{aligned}
\left(\frac{2}{|q|^2+1}q + \frac{|q|^2-1}{|q|^2+1}e_{n+1}\right) \cdot u &= d && \text{by definition of being in } H_d(u) \cap S^n. \\
\frac{2}{|q|^2+1}q \cdot u + \frac{|q|^2-1}{|q|^2+1}e_{n+1} \cdot u &= d && \text{distributing } u. \\
\frac{2}{|q|^2+1}q \cdot u' + \frac{|q|^2-1}{|q|^2+1}u_{n+1} &= d && \text{let } u = u' + (u_{n+1} \cdot e_{n+1})u \\
& && \text{where } u' = (u_1, \ldots, u_n, 0) \\
2q \cdot u' + (|q|^2-1)u_{n+1} &= d(|q|^2+1) && \text{Factoring and multiplying both sides by } |q|^2+1 \\
2q \cdot u' + |q|^2 u_{n+1} - u_{n+1} - d|q|^2 - d &= 0 && \text{expanding terms} \\
2q \cdot u' + |q|^2(u_{n+1} - d) - (u_{n+1} + d) &= 0 && \text{grouping } |q|^2 \text{ terms} \\
\frac{2q \cdot u'}{u_{n+1} - d} + |q|^2 - \frac{u_{n+1}+d}{u_{n+1}-d} &= 0 && \text{dividing by } u_{n+1} - d\text{, provided that } u_{n+1} \neq d \\
\left|q + \frac{u'}{u_{n+1}-d}\right|^2 - \frac{|u'|^2}{(u_{n+1}-d)^2} - \frac{u_{n+1}+d}{u_{n+1}-d} &= 0 && \text{completing squares} \\
\left|q - \frac{u'}{d-u_{n+1}}\right|^2 - \frac{|u'|^2 + (u_{n+1}+d)(u_{n+1}-d)}{(u_{n+1}-d)^2} &= 0 && \text{rearranging terms and adding fractions} \\
\left|q - \frac{u'}{d-u_{n+1}}\right|^2 &= \frac{|u'|^2 + u_{n+1}^2 - d^2}{(u_{n+1}-d)^2} \\
\left|q - \frac{u'}{d-u_{n+1}}\right|^2 &= \frac{1-d^2}{(u_{n+1}-d)^2} && \text{Since } |u'|^2 + u_{n+1}^2 = 1 \text{ (unit vector)}
\end{aligned}$$

Hence, for u a unit vector in $\mathbb{R}^{n+1}$ and $d \in \mathbb{R}$ the set $H_d(u) \cap S^n$ is mapped by σ to a circle of radius:

$$r^2 = \frac{1-d^2}{(u_{n+1}-d)^2} \iff r = \frac{\sqrt{1-d^2}}{|u_{n+1}-d|}$$

And center $\dfrac{u'}{d-u_{n+1}} = \dfrac{1}{d-u_{n+1}}(u_1, \ldots, u_n)$

(Ex. 3)

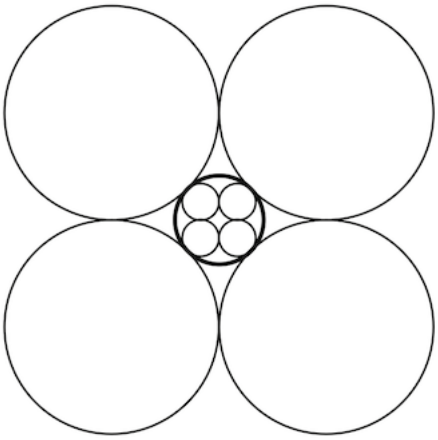

Figure 1

The cube with vertices at $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$ leads to a circle packing of S^2 with 8 circles by taking as centers of the circles the vertices of the cube, and making them so large and equal radius so that they touch. When you stereographically project them into the plane, you get a figure like the one above, where the fat circle in the center is the unit circle. Find the radii and centers of the other projected circles. Hint: First describe the circles as intersections of the sphere with suitable planes, then use the previous exercise.

Solution: Let $p_1 = \dfrac{1}{\sqrt{3}}(1, -1, 1), p_2 = \dfrac{1}{\sqrt{3}}(-1, -1, 1), p_3 = \dfrac{1}{\sqrt{3}}(1, -1, -1), p_4 = \dfrac{1}{\sqrt{3}}(1, 1, 1)$. Denote $p_{i,j}$ as the midpoint between points p_i and p_j for $i \neq j$. Then,

$$p_{1,2} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}(1, -1, 1) + \frac{1}{\sqrt{3}}(-1, -1, 1)\right] = \frac{1}{2\sqrt{3}}(0, -2, 2) = \frac{1}{\sqrt{3}}(0, -1, 1)$$

$$p_{1,4} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}(1, -1, 1) + \frac{1}{\sqrt{3}}(1, 1, 1)\right] = \frac{1}{2\sqrt{3}}(2, 0, 2) = \frac{1}{\sqrt{3}}(1, 0, 1)$$

$$p_{1,3} = \frac{1}{2}\left[\frac{1}{\sqrt{3}}(1, -1, 1) + \frac{1}{\sqrt{3}}(1, -1, -1)\right] = \frac{1}{2\sqrt{3}}(2, -2, 0) = \frac{1}{\sqrt{3}}(1, -1, 0)$$

These points are the midpoints between the corresponding vertices of the cube. Note that these are not unit vectors so they are not on the unit sphere. Therefore, let us normalize these vectors:

$$\widehat{p_{1,2}} = \frac{p_{1,2}}{|p_{1,2}|} = \frac{\frac{1}{\sqrt{3}}(0, -1, 1)}{\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}(0, -1, 1)$$

$$\widehat{p_{1,4}} = \frac{p_{1,4}}{|p_{1,4}|} = \frac{\frac{1}{\sqrt{3}}(1, 0, 1)}{\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}(1, 0, 1)$$

$$\widehat{p_{1,3}} = \frac{p_{1,3}}{|p_{1,3}|} = \frac{\frac{1}{\sqrt{3}}(1, -1, 0)}{\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}(1, -1, 0)$$

These vectors lie on the sphere and through them we can find the plane that describe the circle as intersection with the sphere. Hence, let us find the plane through $\widehat{p_{1,2}}, \widehat{p_{1,4}}$ and $\widehat{p_{1,3}}$:

1. Find the normal vector of the plane: let v and w be two vectors in the plane as follows:

$$v = \widehat{p_{1,4}} - \widehat{p_{1,2}} = \frac{1}{\sqrt{2}}(1, 0, 1) - \frac{1}{\sqrt{2}}(0, -1, 1) = \frac{1}{\sqrt{2}}(1, 1, 0)$$

$$w = \widehat{p_{1,4}} - \widehat{p_{1,3}} = \frac{1}{\sqrt{2}}(1,0,1) - \frac{1}{\sqrt{2}}(1,-1,0) = \frac{1}{\sqrt{2}}(0,1,1)$$

Clearly, the normal vector is $n = (1,-1,1)$. The unit normal vector is then $u = \dfrac{1}{\sqrt{3}}(1,-1,1)$

2. We can find d by finding the equation of the plane:

$$u \cdot [\widehat{p_{1,4}} - (x,y,z)] = 0 \Longrightarrow \frac{1}{\sqrt{3}}(1,-1,1) \cdot \left[\frac{1}{\sqrt{2}}(1,0,1) - (x,y,z)\right] = 0 \iff x - y + z = \frac{2}{\sqrt{2}}$$

Hence $d = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}^2} = \sqrt{2}$

Therefore, our unit normal is $\boxed{u = \dfrac{1}{\sqrt{3}}(1,-1,1)}$ and $\boxed{d = \sqrt{2}}$. Using the previous exercise, we can compute the radius and center of the projected circle:

Center: $\dfrac{1}{d - u_{n+1}}(u_1, \ldots, u_n) = \dfrac{1}{\sqrt{2} - \dfrac{1}{\sqrt{3}}}\left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{\dfrac{\sqrt{6}-1}{\sqrt{3}}}\left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right) = \boxed{\dfrac{1}{\sqrt{6}-1}(1,-1)}$

Radius: $\dfrac{\sqrt{1-d^2}}{|u_{n+1} - d|} = \dfrac{\sqrt{1-\sqrt{2}^2}}{|1/\sqrt{3} - \sqrt{2}|}$

Since all circles are reflections of other circles, the radius is the same and the center is just a reflection, i.e., chaining sings.

(Ex. 4) Show that two circles with centers at $p, q \in \mathbb{R}^2$ and radii $r, s > 0$ intersect at an angle ϕ with

$$cos\phi = \frac{r^2 + s^2 - |p-q|^2}{2rs}$$

This angle of intersection is defined as the angle between the vectors $x - p$ and $x - q$ for a point x that lies on both circles. Hint: expand $|(x-p) - (x-q)|^2$.

Solution: First, using the definition of dot product, if we have vectors a and b, then $a \cdot b = |a||b|cos\theta$, where θ is the angle between the vectors a and b. We will use this formula to find the angle of intersection of two circles.

To begin, let us use the hint and expand $|(x-p) - (x-q)|^2$:

$$|(x-p)-(x-q)|^2 = |(x-p)|^2 - 2(x-p)\cdot(x-q) + |(x-q)|^2 \Longrightarrow (x-p)\cdot(x-q) = \frac{|(x-p)|^2 + |(x-q)|^2 - |(x-p)-(x-q)|^2}{2}$$

We can simplify this formula quite a bit. Note that by constructing the vector $x - p$ must have length r, since this is a vector from the center of the first circle to the intersection of the circles. Hence, $|x - p| = r$. Likewise, $|x - q| = s$. Also, $|(x-p) - (x-q)| = |q - p| = |p - q|$, since the vector $q - p$ has the same length as the vector $p - q$ (indeed, $p - q = -1(q - p)$). Replacing in our identity above:

$$(x-p)\cdot(x-q) = \frac{|(x-p)|^2 + |(x-q)|^2 - |(x-p)-(x-q)|^2}{2} = \frac{r^2 + s^2 - |p-q|^2}{2}$$

Finally, replace this in the dot product formula for the vectors $x - p$ and $x - q$:

$$(x-p)\cdot(x-q) = |x-p||x-q|cos\phi \Longrightarrow cos\phi = \frac{(x-p)\cdot(x-q)}{|x-p||x-q|} = \frac{\frac{r^2+s^2-|p-q|^2}{2}}{rs} = \frac{r^2+s^2-|p-q|^2}{2rs}$$

Establishing the result.

(Ex. 5) Show that the dihedral angle of the regular Euclidean octahedron is approximately 109.47°.

Solution: Consider an octahedron centered at $(0,0,0)$ with vertices at $(\pm 1,0,0), (0,\pm 1,0)$ and $(0,0,\pm 1)$. To compute the dihedral angle is the same as computing the angle between normal vectors to two adjacent faces. Hence, let us first find normal to two adjacent faces of this octahedron:

(a) Face on the first octant: two edges are given by the vectors v_1 and v_2, where:

$$v_1 = (0,1,0) - (1,0,0) = (-1,1,0)$$

$$v_2 = (0,0,1) - (1,0,0) = (-1,0,1)$$

From this we get the first normal $n_1 = (1,1,1)$

(b) Face on the fourth octant: again, two edges are given by the vector t_1 and t_2, where

$$t_1 = (0,-1,0) - (1,0,0) = (-1,-1,0)$$

$$t_2 = (0,0,1) - (1,0,0) = (-1,0,1)$$

From this we get the second normal $n_2 = (-1,1,-1)$

Since these are adjacent faces, we can compute the dihedral angle by the dot product between n_1 and n_2:

$$cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} \Longrightarrow cos\theta = \frac{(1,1,1)\cdot(-1,1,-1)}{\sqrt{3}\sqrt{3}} = \frac{-1+1-1}{3} = -\frac{1}{3} \Longrightarrow cos\theta = -\frac{1}{3} \Longrightarrow \theta = arccos(-\frac{1}{3}) \Longrightarrow \boxed{\theta \approx 109.47°}$$

Since the angles of the regular Euclidean octahedron are scale invariant, the above argument works for an octahedron of any size.