— M436 — Homework Assignment 7 —

Due: Wednesday, October 22, in class. Each problem is worth 20 points. Please show all your work. Homework that is illegible or discourages the reader otherwise from looking at it will be returned ungraded.

Exercise 1

Recall that an isometry of a metric space X is a transformation $\phi : X \to X$ such that $|\phi(x) - \phi(y)| = |x - y|$ for all points $x, y \in X$. A similarity with dilation $\lambda > 0$ is a transformation $\phi : X \to X$ such that $|\phi(x) - \phi(y)| = \lambda|x - y|$. Show for given metric space:

1. Similarities are invertible.
2. For two similarities, the dilatation of $\phi \circ \psi$ is the product of the dilatations of ϕ and ψ.
3. The set of all similarities $\phi : X \to X$ forms a group under composition.
4. The set of all isometries $\phi : X \to X$ forms a group under composition.

Exercise 2

The **figure 1** shows a partial tiling of the plane by equilateral triangles, squares, and regular hexagons. The black triangle has its vertices at the centers of the respective polygons, so it is a $30 - 60 - 90$ degree triangle. The reflections about its edges are denoted by α, β,and γ. The transformations $\alpha \circ \beta$, $\beta \circ \gamma$, and$\gamma \circ \alpha$ are all rotations about the center of some polygon by some angle. Determine the polygons and the angles. The transformations ϕ, ψ, σ are rotations by 120°, 180°, 60° about the centers of the indicated triangle, square, hexagon. Show that $\sigma = \beta \circ \gamma \circ \alpha \circ \beta$, and find compositions of α, β,and γ that equal ϕ and ψ.

Exercise 3

In the taxicab geometry of $\mathbf{Z}^2$, find a formula for the number of shortest taxicab paths from $(0, 0)$ to (a, b).

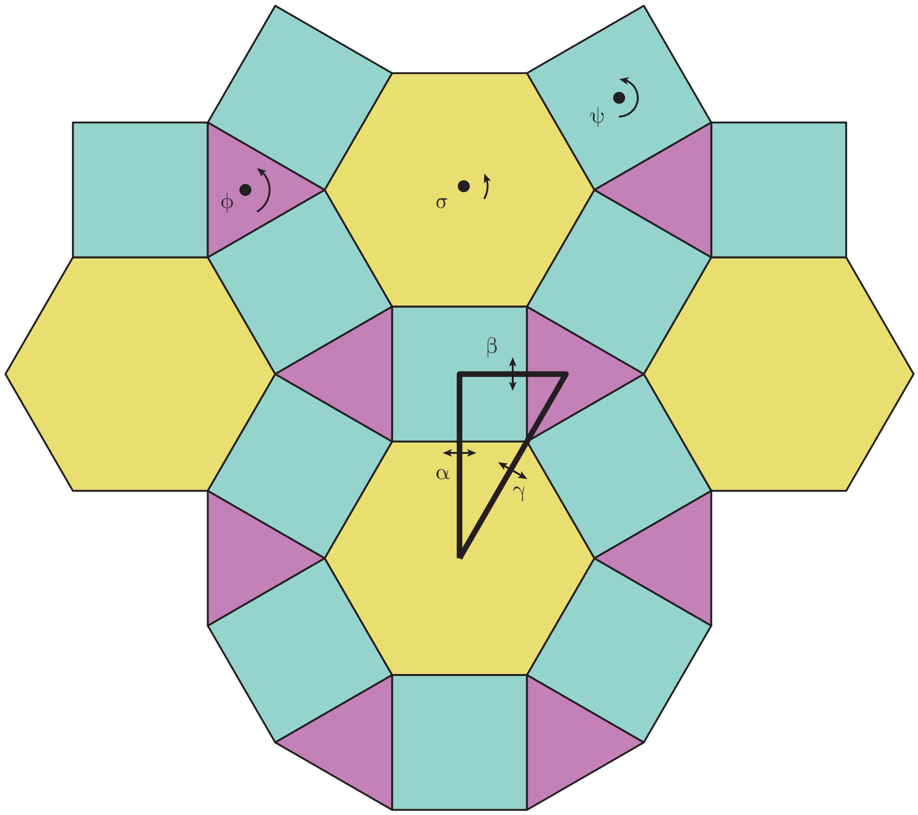

Figure 1 Reflecting about reflections

Exercise 4

In the taxicab geometry of $\mathbf{R}^2$, determine the shape of the ellipse that has distance sum 12 to the two focal points $p = (1, 2)$ and $q = (-1, -2)$. What conditions must the focal points and focal distance satisfy in order that a taxicab ellipse is a hexagon?

Exercise 5

In $\mathbf{F}_2^n$ with the Hamming distance as metric, find a formula for the number of points in the sphere of radius r, for any integer $1 \leq r \leq n$.