M436 - Introduction to Geometries - Homework 7

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(Ex. 1) Recall that an isometry of a metric space X is a transformation $\phi : X \to X$ such that $|\phi(x) - \phi(y)| = |x - y|$ for all points $x, y \in X$. A similarity with dilation $\lambda > 0$ is a transformation $\phi : X \to X$ such that $|\phi(x) - \phi(y)| = \lambda|x - y|$. Show for given metric space:

1. For two similarities, the dilatation $\phi \circ \psi$ is the product of dilatations of ϕ and ψ.
2. The set of all similarities $\phi : X \to X$ forms a group under composition.
3. The set of all isometries $\phi : X \to X$ forms a group under composition.

Solution:

1. Let ϕ, ψ be two similarities with factors $\lambda_\phi > 0$ and $\lambda_\psi > 0$ respectively.
   To show that the composition of similarities is a similarity we need to show that for every $x, y \in X$
   $$|\phi \circ \psi(x) - \phi \circ \psi(y)| = |\phi(\psi(x)) - \phi(\psi(y))| \stackrel{?}{=} \lambda|x - y| \qquad \text{for some } \lambda \in \mathbb{R}$$
   $$\begin{aligned} |\phi(\psi(x)) - \phi(\psi(y))| &= \lambda_\phi|\psi(x) - \psi(y)| && \text{since } \phi \text{ is a similarity with factor } \lambda_\phi \in \mathbb{R} \\ &= \lambda_\phi(\lambda_\psi|x - y|) && \text{since } \psi \text{ is a similarity with factor } \lambda_\psi \in \mathbb{R} \\ &= (\lambda_\phi\lambda_\psi)|x - y| && \text{rearranging factors} \end{aligned}$$
   Therefore, $|\phi(\psi(x)) - \phi(\psi(y))| = (\lambda_\phi\lambda_\psi)|x - y|$ which shows that the composition of similarities is a similarity where the factor is the product of each individual factor.
2. In part 1. it was shown that similarities are closed under composition. To show that a similarity is invertible we will add the extra condition that the maps under considerations are surjective. With this extra condition, it suffices then to show that similarities are injective to conclude that they are invertible.
   To show injectivity, let ϕ be a similarity with factor $\lambda_\phi \in \mathbb{R}$ and $x, y \in X$. Suppose $\phi(x) = \phi(y)$. Then, since ϕ is a similarity we must have: $|\phi(x) - \phi(y)| = \lambda_\phi|x - y|$, but $|\phi(x) - \phi(y)| = 0$ by axioms of the distance metric. Hence, $0 = \lambda_\phi|x - y|$, which means that $\lambda_\phi = 0$ or $|x - y| = 0$, but λ_ϕ we assume to be a positive number, so it must follow that $|x - y| = 0$ and hence $x = y$. This shows that similarities are injective.
   Again, we assumed surjectivity and proved injectivity. Both of these imply that our maps under consideration are bijective. So we have inverses. To check for group structure we cheek the following:
   i. Closed under composition: checked in 1.
   ii. Associative: we showed in previous homeworks that function composition is associative. This is a special case of that and so its implied by it.
   iii. Existence of Identity: take the identity map $\phi(x) = x$. Clearly distances are preserve and so it is a similarity with factor 1. Another way of viewing this is that the group of similarities is a subgroup of the groups of invertible functions from a set to itself and hence, the identity must be the same, i.e., the identity map.
   iv. Existence of Inverses: already checked above.
3. An isometry is a special case of a similarity where the factor is just 1. So isometries form a subgroup of the group of similarities and hence are a group. Again, taking into consideration our hypothesis that maps are surjective.

(Ex. 2) The figure 1 shows a partial tiling of the plane by equilateral triangles, squares, and regular hexagons. The black triangle has its vertices at the centers of the respective polygons, so it is a $30 - 60 - 90$ degree triangle. The reflections about its edges are denoted by α, β and γ. The transformations $\alpha \circ \beta, \beta \circ \gamma$, and$\gamma \circ \alpha$ are all rotations about the center of some polygon by some angle. Determine the polygons and the angles. The transformations ϕ, ψ, σ are rotations by $120°, 180°, 60°$ about the centers of the indicated triangle, square, hexagon. Show that $\sigma = \beta \circ \gamma \circ \alpha \circ \beta$, and find compositions of α, β,and γ that equal ϕ and ψ.

Solution: The transformation $\alpha \circ \beta$ corresponds to a rotation about the center square by $180°$.
The transformation $\beta \circ \gamma$ corresponds to a rotation about the triangle by $120°$.
The transformation $\gamma \circ \alpha$ corresponds to a rotation about the hexagon by $60°$.
$\sigma = \beta \circ \gamma \circ \alpha \circ \beta$ and $\phi = \alpha \circ \gamma \circ \beta \circ \alpha$. But then ψ is just the composition

$$\psi = \phi \circ \sigma = (\alpha \circ \gamma \circ \beta \circ \alpha) \circ (\beta \circ \gamma \circ \alpha \circ \beta)$$

which, by the first part of the exercise, can be read as: perform a rotation about the center square by $180°$, then a rotation about the triangle by $120°$, then rotation about the diagonal left square by $180°$ and finally a a rotation about the hexagon by $60°$, yielding the desired rotation.

(Ex. 3) In the taxicab geometry of $\mathbb{Z}^2$, find a formula for the number of shortest taxicab paths from $(0,0)$ to (a,b)

Solution: First, let us develop a formula in the case where both a and b are positive numbers. Let $\#(a,b)$ be the number of shortest paths from $(0,0)$ to (a,b). I claim that $\#(a,b)$ is given by:

$$\#(a,b) = \binom{a+b}{a} \qquad \text{or equivalently} \qquad \#(a,b) = \binom{a+b}{b}$$

We can prove this result by induction. First, let us check a few base cases:

1. For pairs of the form $(0,b)$ and $(a,0)$ the number of shortest paths from $(0,0)$ to these is just 1, corresponding to the horizontal of vertical straight path respectively. This corresponds to $\binom{0+b}{b} = \binom{a+0}{0} = 1$
2. For $(a,b) = (1,1)$ we have that $\#(1,1) = \binom{1+1}{1} = \binom{2}{1} = 2$, which is true.
3. For $(a,b) = (1,2)$ we have that $\#(1,2) = \binom{1+2}{2} = \binom{3}{2} = 3$, which is true and the same as $\#(2,1) = \binom{2+1}{1} = 3$.

For the inductive step, suppose $\#(n,m) = \binom{n+m}{n}$ is true for all pairs (n,m) where $1 \le n \le a$ or $1 \le m \le b$. We want to show the following three statements:

(a) $\#(a+1,b+1) \stackrel{?}{=} \binom{(a+1)+(b+1)}{b+1}$. Clearly, $\#(a+1,b+1)$ is the sum of $\#(a,b+1)$ and $\#(a+1,b)$. This is because a shortest path to $(a+1,b+1)$ can go either through $\#(a,b+1)$ or $\#(a+1,b)$, but once it is in one of those places, there is no choice but to take one more step to complete said path. Now, using the inductive hypothesis on $\#(a,b+1)$ and $\#(a+1,b)$:

$$\begin{aligned}
\#(a+1,b+1) &= \#(a,b+1) + \#(a+1,b) && \text{as explained before} \\
&= \tbinom{a+b+1}{b+1} + \tbinom{a+1+b}{b} && \text{by inductive hypothesis} \\
&= \frac{(a+b+1)!}{a!(b+1)!} + \frac{(a+1+b)!}{(a+1)!b!} && \text{by definition of binomial coefficient} \\
&= \frac{(a+1)!b!(a+b+1)! + a!(b+1)!(a+1+b)!}{a!(b+1)!(a+1)!b!} && \text{adding fractions} \\
&= \frac{(a+b+1)![(a+1)!b! + a!(b+1)!]}{a!(b+1)!(a+1)!b!} && \text{factoring } (a+b+1)! \\
&= \frac{(a+b+1)![a!b![(a+1)+(b+1)]]}{a!(b+1)!(a+1)!b!} && \text{further factoring} \\
&= \frac{(a+b+1)![(a+b+1)+1]}{(b+1)!(a+1)!} && \text{canceling equal terms, rearranging} \\
&= \frac{(a+1+b+1)!}{(b+1)!(a+1)!} && \text{by definition of factorial} \\
&= \tbinom{(a+1)+(b+1)}{b+1} && \text{by definition of binomial coefficient}
\end{aligned}$$

(b) $\#(a,b+1) \stackrel{?}{=} \binom{(a)+(b+1)}{b+1}$. This statement is very similar to (a) so I won't repeat the argument here.

(c) $\#(a+1,b) \stackrel{?}{=} \binom{(a+1)+(b)}{b}$. Same as (b).

Finally, by symmetry of $\mathbb{Z}^2$, we know that the number of shortest path when considering negative numbers will be identical as translating the point to all positive coordinates. Therefore, the correct formula is given by:

$$\boxed{\binom{|a|+|b|}{|a|} \qquad \text{or equivalently} \qquad \binom{|a|+|b|}{|b|}}$$

(Ex. 4) In the taxicab geometry of $\mathbb{R}^2$, determine the shape of the ellipse that has distance sum 12 to the two focal points $p = (1, 2)$ and $q = (-1, -2)$. What conditions must the focal points and focal distance satisfy in order that a taxicab ellipse is a hexagon?

Solution: Let $E = \{(x, y) : d_1((x, y), (1, 2)) + d_1((x, y), (-1, -2)) = 12\}$, where d_1 is the distance of the taxicab geometry, i.e., $d_1((x, y), (1, 2)) = |x - 1| + |x - 2|$, so our ellipse becomes:

$$E = \{(x, y) : |x - 1| + |y - 2| + |x + 1| + |y + 2| = 12\}$$

There are a few cases to consider:

$x \geq 1, y \geq 2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow x-1+y-2+x+1+y+2 = 12 \Longrightarrow \boxed{y = 6 - x}$

$x \geq 1, -2 \leq y \leq 2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow x-1+2-y+x+1+y+2 = 12 \Longrightarrow \boxed{x = 4}$

$-1 \leq x \leq 1, y \geq 2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow 1-x+y-2+x+1+y+2 = 12 \Longrightarrow \boxed{y = 5}$

$x \geq 1, y \leq -2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow x-1+2-y+x+1-2-y = 12 \Longrightarrow \boxed{y = x - 6}$

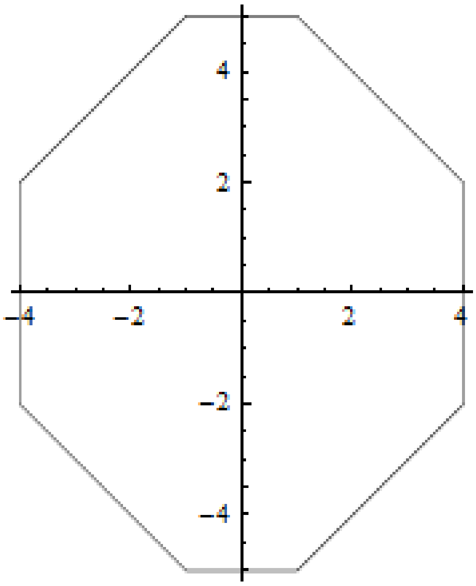
$0 \leq x \leq 1, y \leq -2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow 1-x+2-y+x+1-2-y = 12 \Longrightarrow \boxed{y = -5}$

$x \leq -1, y \leq -2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow 1-x+2-y-1-x-2-y = 12 \Longrightarrow \boxed{y = -x - 6}$

$x \leq -1, -2 \leq y \leq 0$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow 1-x+2-y-1-x+y+2 = 12 \Longrightarrow \boxed{x = -4}$

$x \leq -1, y \geq 2$, then $|x-1|+|y-2|+|x+1|+|y+2| = 12 \Longrightarrow 1-x+y-2-1-x+y+2 = 12 \Longrightarrow \boxed{y = x + 6}$

Plotting all these line segments we obtained the following octagon:



Claim: Consider the ellipse: $E = \{(x, y) : |x - x_1| + |y - y_1| + |x - x_2| + |y - y_2| = c\}$. In order for E to be a hexagon the focal points must lie on a horizontal or vertical line and the focal distance must satisfy $d_1((x_1, y_1), (x_2, y_2)) < c$ but not zero.

Proof (idea): First note that if $d_1((x_1, y_1), (x_2, y_2)) = 0$ we get a circle so we do not include this case. Also, if $d_1((x_1, y_1), (x_2, y_2)) = c$, then we get a square. Hence, we assume $d_1((x_1, y_1), (x_2, y_2)) < c$.

Suppose that the focal points lie on a vertical line in the $y-$axis. This is enough to show both cases because a simple rotation would give you the horizontal case. Hence, without loss of generality $(x_1, y_1) = (0, y_1)$ and $(x_2, y_2) = (0, y_2)$. Replacing in our definition of E:

$$E = \{(x, y) : |x - x_1| + |y - y_1| + |x - x_2| + |y - y_2| = c\} \Longrightarrow E = \{(x, y) : |x| + |y - y_1| + |x| + |y - y_2| = c\}$$

$$\Longleftrightarrow E = \{(x, y) : 2|x| + |y - y_1| + |y - y_2| = c\}$$

Now we can analyze the equation: $2|x| + |y - y_1| + |y - y_2| = c$

$x \geq 0, y \geq y_1, y \geq y_2$, then $2x + y - y_1 + y - y_2 = c \Longrightarrow 2x + y = d$, for some constant d.
$x \geq 0, y \leq y_1, y \geq y_2$, then $2x + y_1 - y + y - y_2 = c \Longrightarrow x = d$, for some constant d.
$x \geq 0, y \geq y_1, y \leq y_2$, then $2x + y - y_1 + y_2 - y = c \Longrightarrow x = d$, for some constant d.
$x \geq 0, y \leq y_1, y \leq y_2$, then $2x + y_1 - y + y_2 - y = c \Longrightarrow 2x - 2y = d$, for some constant d.
$x \leq 0, y \geq y_1, y \geq y_2$, then $-2x + y - y_1 + y - y_2 = c \Longrightarrow -2x + y = d$, for some constant d.
$x \leq 0, y \leq y_1, y \leq y_2$, then $-2x + y_1 - y + y_2 - y = c \Longrightarrow -2x - y = d$, for some constant d.

All other cases are non-satisfiable and result in no points. Hence, E is an hexagon.

(Ex. 5) In $\mathbb{F}_2^n$ with the Hamming distance as metric, find a formula for the number of points in the sphere of radius r, for any integer $1 \leq r \leq n$.

Solution: First note that the $|\mathbb{F}_2^n| = 2 \cdot 2 \cdots 2 = 2^n$. Now, if we fix a point $p \in \mathbb{F}_2^n$ and a radius r such that $1 \leq r \leq n$ and we want to find how many points are there in the sphere of radius r centered at p using the Hamming distance as metric, then we are essentially asking how many points differ from our point p in up to r positions. We can begin to count these points by cases: there is always exactly one point that differ in no position at all from p and that is p itself. If we want the points that differ in exactly two positions, then we can choose $\binom{n}{2}$ places to change. We do not care about the order in which we choose places, so that we uses a combination. Likewise, for differences of exactly three positions, we can choose $\binom{n}{3}$ places to change. Keep this count up to r, so that the number of points is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{r-1} + \binom{n}{r}$. We can write this as:

$$\boxed{\text{number of points in the sphere of radius } r = \sum_{i=0}^{r} \binom{n}{i}}$$

Note that this formula makes sense in the borderline cases: if $r = 0$ then $\sum\limits_{i=0}^{r} \binom{n}{i} = \binom{n}{0} = 1$, and indeed there is only one point in the sphere of radius 0, i.e., the center itself. Also, if $r = n$ then $\sum\limits_{i=0}^{n} \binom{n}{i} = 2^n$, so that all points in the space are contained in an sphere of maximum radius.