M436 - Introduction to Geometries - Homework 6

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(Ex. 1) Consider in $\mathbb{R}P^2$ the triangle with vertices $p_1 = (-3,-4), p_2 = (3,-4)$, and $p_3 = (-3,4)$. Determine the centers and radii of the circumcircle and incircle of this triangle. The point $q_1 = (-4,-3)$ lies also on the circumcircle. Find the coordinates of the other two vertices q_2 and q_3 of the Poncelet triangle determined by q_1 and the two circles

Solution: Let a be the length of the side of the triangle corresponding to the line segment from $(-3,4)$ to $(3,-4)$. Then

$$a = d((-3,4),(3,-4)) = \sqrt{(-3-3)^2 + (4+4)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Likewise, let b be the length of the side of the triangle corresponding to the line segment from $(-3,-4)$ to $(3,-4)$. Then

$$b = d((-3,-4),(3,-4)) = \sqrt{(-3-3)^2 + (-4+4)^2} = \sqrt{6^2} = 6$$

Lastly, let c be the length of the side of the triangle corresponding to the line segment from $(-3,-4)$ to $(-3,4)$. Then

$$c = d((-3,-4),(-3,4)) = \sqrt{(-3+3)^2 + (-4-4)^2} = \sqrt{8^2} = 8$$

The center (x_I, y_I) of the incircle is given by the formula:

$$(x_I, y_I) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where $(x_1,y_1) = (-3,-4), (x_2,y_2) = (-3,4)$ and $(x_3,y_3) = (3,-4)$, i.e., the vertices of the triangle. Hence:

$$(x_I, y_I) = \left(\frac{10(-3)+6(-3)+8(3)}{10+8+6}, \frac{10(-4)+6(4)+8(-4)}{10+8+6}\right) = \left(\frac{-30-18+24}{24}, \frac{-40+24-32}{24}\right) = \left(\frac{-24}{24}, \frac{-48}{24}\right)$$

So the center of the incircle is $\boxed{(-1,-2)}$.The radius of the incircle r_I is given by:

$$r_I = \frac{2 \cdot \text{area of triangle}}{\text{sum of sides}}$$

In our case, the area of the triangle is $(b \cdot h)/2 = (6 \cdot 8)/2 = 24$, and the sum of the sides is $8 + 6 + 10 = 24$. Hence:

$$\boxed{r_I = \frac{2 \cdot 24}{24} = 2}$$

The radius of the circumcircle is given by the formula $R_C = \dfrac{a \cdot b \cdot c}{4 \cdot \text{ area}}$, so we can solve:

$$\boxed{R_C = \frac{10 \cdot 6 \cdot 8}{4 \cdot 24} = 5}$$

Since the radius is 5, we must have that the center $\boxed{(x_C, y_C) = (0,0)}$, so that the circumcircle meets with every vertex. We were given the other two points of the Poncelet triangle: $q_2 = (1, 2\sqrt{6})$ and $q_3 = (1, -2\sqrt{6})$. Clearly the line through $q_1 q_2$ given by $y = \frac{1}{5}\left(2\sqrt{6}+3\right)x + \frac{1}{5}\left(8\sqrt{6}-3\right)$ intersects the circle in only one point since this is a linear variable. Likewise, the line through $q_1 q_3$ and $q_2 q_3$ will also intersect the circle in only one point, showing that q_1, q_2 and q_3 form a Poncelet triangle.

(Ex. 2) Find a projective linear transformation of $\mathbb{R}P^2$ that maps the conic $x^2 + y^2 = z^2$ to the conic $xz = y^2$. Verify your claim

Solution: First note that $x^2 + y^2 = z^2 \iff y^2 = z^2 - x^2 \iff y^2 = (z-x)(z+x)$. Now consider the linear change:

$$z - x \mapsto x, \quad y \mapsto y, \quad z + x \mapsto z$$

Applying this change we get that $y^2 = (z-x)(z+x) \implies y^2 = xz$, verifying that the above projective linear transformation maps the conic $x^2 + y^2 = z^2$ to the conic $xz = y^2$

(Ex. 3) Find a conic in the projective plane $\mathbb{F}_5P^2$ that contains the points $(1:1:2),(1:1:3),(1:3:0),(1:4:2)$ and $(1:4:3)$. Find one more point on this conic.

Solution: By Theorem 1 in page 80, since no three of the given points are collinear, we can find the unique conic containing these five points. First, let us label the points as follows: $p_1=(1:1:2), p_2=(1:1:3), p_3=(1:3:0), p_4=(1:4:2)$ and $p_5=(1:4:3)$. Now, we can find two degenerate conics containing the lines p_1p_2, p_3p_4 and p_1p_3, p_2p_4 respectively. Call these A and B. Then,

$$A=(p_1\times p_2)\cdot(p_3\times p_4)^T=\left[\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\3\end{pmatrix}\right]\cdot\left[\begin{pmatrix}1\\3\\0\end{pmatrix}\times\begin{pmatrix}1\\4\\2\end{pmatrix}\right]^T=\begin{pmatrix}1\\4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}^T=\begin{pmatrix}1&3&1\\4&2&4\\0&0&0\end{pmatrix}$$

$$B=(p_1\times p_3)\cdot(p_2\times p_4)^T=\left[\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\0\end{pmatrix}\right]\cdot\left[\begin{pmatrix}1\\1\\3\end{pmatrix}\times\begin{pmatrix}1\\4\\2\end{pmatrix}\right]^T=\begin{pmatrix}4\\2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\3\end{pmatrix}^T=\begin{pmatrix}0&4&2\\0&2&1\\0&2&1\end{pmatrix}$$

Finally, solve for t in the equation $p_5^T(A+tB)p_5=0$

$$\begin{pmatrix}1&4&3\end{pmatrix}\left\{\begin{bmatrix}1&3&1\\4&2&4\\0&0&0\end{bmatrix}+t\begin{bmatrix}0&4&2\\0&2&1\\0&2&1\end{bmatrix}\right\}\begin{pmatrix}1\\4\\3\end{pmatrix}=0$$

$$\begin{pmatrix}1&4&3\end{pmatrix}\begin{bmatrix}1&3+4t&1+2t\\4&2+2t&4+t\\0&2t&t\end{bmatrix}\begin{pmatrix}1\\4\\3\end{pmatrix}=0$$

$$\begin{pmatrix}1&4&3\end{pmatrix}\begin{bmatrix}1+2+t+3+t\\4+3+3t+2+3t\\3t+3t\end{bmatrix}=0$$

$$\begin{pmatrix}1&4&3\end{pmatrix}\begin{bmatrix}2t+1\\t+4\\t\end{bmatrix}=0$$

$$2t+1+4t+1+3t=0\iff 4t+2=0\iff 4t=3\iff \boxed{t=2}$$

Therefore, the conic containing all five points is given by :

$$A+2B=\begin{bmatrix}1&3&1\\4&2&4\\0&0&0\end{bmatrix}+2\begin{bmatrix}0&4&2\\0&2&1\\0&2&1\end{bmatrix}=\boxed{\begin{bmatrix}1&1&0\\4&1&1\\0&4&2\end{bmatrix}}$$

Indeed it is true that $p_i^T(A+2B)p_i=0$ for $1\le i\le 5$.

Note that the point $(x:y:z)$ lies on the conic if and only if:

$$\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}1&1&0\\4&1&1\\0&4&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\iff\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}x+y\\4x+y+z\\4y+2z\end{pmatrix}=0\iff x^2+xy+4xy+y^2+yz+4yz+2z^2=0\iff$$

$$x^2+y^2+2z^2=0$$

Note that this conic is symmetric in x and y, i.e., if $(x:y:z)$ is on the conic then $x^2+y^2+2z^2=0\iff y^2+x^2+2z^2=0\iff (y:x:z)$ is on the conic. Since we know that the point $(1:4:2)$ is on the conic, we conclude that the point $(4:1:2)$ is on the conic. Indeed:

$$4^2+1^2+2(2^2)=16+1+8=25\equiv 0\pmod 5$$

(Ex. 4) For which value of t is the conic $2x^2 - 2y^2 - tz^2 + 3xy - xz + 3yz = 0$ in $\mathbb{R}P^2$ degenerate, and in which two lines does it decompose in this case?

Solution: First, let us find the symmetric matrix that defines this conic. I claim it is this matrix:

$$C = \begin{pmatrix} 2 & 3/2 & -1/2 \\ 3/2 & -2 & 3/2 \\ -1/2 & 3/2 & -t \end{pmatrix}$$

We can check that the conic is the set $\{x \in \mathbb{R}P^2 : x^T C x = 0\}$ by computing:

$$0 = x^T C x = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 3/2 & -1/2 \\ 3/2 & -2 & 3/2 \\ -1/2 & 3/2 & -t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2x + \frac{3}{2}y - \frac{1}{2}z \\ \\ \frac{3}{2}x - 2y + \frac{3}{2}z \\ \\ -\frac{1}{2}x + \frac{3}{2}y - tz \end{pmatrix} =$$

$$2x^2 + \frac{3}{2}xy - \frac{1}{2}xz + \frac{3}{2}xy - 2y^2 + \frac{3}{2}yz - \frac{1}{2}xz + \frac{3}{2}yz - tz^2 = 2x^2 - 2y^2 - tz^2 + \left[\frac{3}{2} + \frac{3}{2}\right]xy - \left[\frac{1}{2} + \frac{1}{2}\right]xz + \left[\frac{3}{2} + \frac{3}{2}yz\right] =$$

$$2x^2 - 2y^2 - tz^2 + 3xy - xz + 3yz = 0$$

To find the value of t for which the conic is degenerate, we need to find t for which the determinant of C is zero:

$$\begin{aligned} det(C) = det\left|\begin{pmatrix} 2 & 3/2 & -1/2 \\ 3/2 & -2 & 3/2 \\ -1/2 & 3/2 & -t \end{pmatrix}\right| &= 2\begin{pmatrix} -2 & 3/2 \\ 3/2 & -t \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3/2 & 3/2 \\ -1/2 & -t \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3/2 & -2 \\ -1/2 & 3/2 \end{pmatrix} \\ &= 2\left(2t - \frac{9}{4}\right) - \frac{3}{2}\left(-\frac{3}{2}t + \frac{3}{4}\right) - \frac{1}{2}\left(\frac{9}{4} - 1\right) \\ &= 4t - \frac{9}{2} + \frac{9}{4}t - \frac{9}{8} - \frac{9}{8} + \frac{1}{2} \\ &= 4t + \frac{9}{4}t + \left[\frac{1}{2} - \frac{9}{2} - \frac{9}{8} - \frac{9}{8}\right] \\ &= \frac{25}{4}t - \frac{25}{4} \end{aligned}$$

Hence, for the value of t: $\frac{25}{4}t - \frac{25}{4} = 0 \Longrightarrow \boxed{t = 1}$, we have that the conic is degenerate. In this case the conic is:

$$2x^2 - 2y^2 - z^2 + 3xy - xz + 3yz = 0 \iff (x + 2y - z)(2x - y + z) = 0$$

That is, the conic decompose in two lines given in homogeneous coordinates by $\boxed{(1 : 2 : -1) \text{ and } (2 : -1 : 1)}$.

(Ex. 5) In $\mathbb{R}P^2$, find the two tangents to the conic $3x^2 - y^2 + z^2 + 2xz = 0$ that pass through the point $(1 : 1 : -2)$. Find also the points where these lines touch the conic.

Solution: The conic $3x^2 - y^2 + z^2 + 2xz = 0$ is given by the symmetric matrix $C = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ because:

$$0 = x^T C x = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3x + z \\ \\ -y \\ \\ x + z \end{pmatrix} = 3x^2 + xz - y^2 + xz + z^2 = 3x^2 - y^2 + z^2 + 2xz$$

Let $p = (1 : 1 : -2)$ and $q = (x : y : z)$ be a point in the conic. Then, the following two conditions hold:

(i) $\quad q^T A q = 0 \quad$ since q is a point on the conic A

(ii) $\quad p^T(Aq) = 0 \quad$ since the point p must be incident with line Aq, where Aq is the tangent line we want to find

From condition (ii) we get that:

$$p^T(Aq) = 0 \iff 0 = p^T\left[\begin{pmatrix}3 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right] \iff 0 = (1 \quad 1 \quad -2)\begin{pmatrix}3x+z\\ -y\\ x+z\end{pmatrix} \iff 0 = 3x+z-y-2x-2z = x-y-z$$

Therefore, we have that $z = x - y$. Replacing this into condition (i), which is equivalent to replacing into the equation for the conic:

$$3x^2 - y^2 + z^2 + 2xz = 0 \Longrightarrow 3x^2 - y^2 + (x-y)^2 + 2x(x-y) = 0 \Longrightarrow 3x^2 - y^2 + x^2 - 2xy + y^2 + 2x^2 - 2xy = 0$$

$$\Longrightarrow 6x^2 - 4xy = 0 \iff 3x^2 - 2xy = 0 \iff x(3x-2y) = 0$$

From which we obtain the two solutions:

$x = 0$: , then we can solve for z in terms of y, i.e., $z = x - y = -y$. The form of a general point is then $(0 : y : -y)$, from which we can pick a representative, say $\boxed{(0:1:-1)}$

$3x - 2y = 0$:, then $x = \frac{2}{3}y$. We can solve for z, i.e., $z = x - y = \frac{2}{3}y - y = -\frac{1}{3}y$. The form of a general point is given by $(\frac{2}{3}y : y : -\frac{1}{3}y) = (2y : 3y : -y)$, from which we can pick a representative, say $\boxed{(2:3:-1)}$

So the points where the tangent lines that pass through the point $(1 : 1 : -2)$ touch the conic are:

$$\boxed{q_1 = (0:1:-1)} \text{ and } \boxed{q_2 = (2:3:-1)}$$

Finally, we can find the tangent lines q_1p and q_2p by taking the cross product:

$$q_1p = q_1 \times p = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix} \times \begin{pmatrix}1\\ 1\\ -2\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

$$q_2p = q_2 \times p = \begin{pmatrix}2\\ 3\\ -1\end{pmatrix} \times \begin{pmatrix}1\\ 1\\ -2\end{pmatrix} = \begin{pmatrix}-5\\ 3\\ -1\end{pmatrix}$$