M436 - Introduction to Geometries - Homework 4

Enrique Areyan
September 26, 2014

(Ex. 1) Let $D_1 v = 2\left(v - \binom{1}{1}\right) + \binom{1}{1}$ be a dilation in $\mathbb{R}^2$. Find another dilation $D_2 v = \lambda(v-p)+p$ such that $(D_2 \circ D_1)v = v + \binom{1}{0}$.

Solution: First note that we can write D_1 differently as $D_1(v) = 2\left(v - \binom{1}{1}\right) + \binom{1}{1} = 2v - \binom{2}{2} + \binom{1}{1} = 2v - \binom{1}{1}$. Now,

$$(D_2 \circ D_1)(v) = D_2(D_1(v)) = D_2\left[2v - \begin{pmatrix}1\\1\end{pmatrix}\right] = v + \begin{pmatrix}1\\0\end{pmatrix}$$

By inspection we can deduce that $\lambda = 1/2$, since the coefficient of v is 2 and we want it to be 1. This observation reduces our computation to:

$$D_2\left[2v - \begin{pmatrix}1\\1\end{pmatrix}\right] = \frac{1}{2}\left[2v - \begin{pmatrix}1\\1\end{pmatrix}\right] + \frac{1}{2}p = v - \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}p =$$

Letting $p = \binom{p_1}{p_2}$

$$= v - \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}p = v - \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}p_1\\p_2\end{pmatrix} = v + \frac{1}{2}\left[\begin{pmatrix}p_1\\p_2\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right]$$

Therefore,

$$\frac{1}{2}\left[\begin{pmatrix}p_1\\p_2\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right] = \begin{pmatrix}1\\0\end{pmatrix} \Longrightarrow \frac{1}{2}p_1 - \frac{1}{2} = 1 \text{ and } \frac{1}{2}p_2 - \frac{1}{2} = 0 \Longrightarrow p_1 = 3 \text{ and } p_2 = 1$$

Our dilation D_2 is given by $\boxed{D_2(v) = \frac{1}{2}\left(v - \begin{pmatrix}3\\1\end{pmatrix}\right) + \begin{pmatrix}3\\1\end{pmatrix}}$. We can check that indeed this is the case:

$$\begin{aligned}
(D_2 \circ D_1)(v) = D_2(D_1(v)) = D_2\left[2\left(v - \tbinom{1}{1}\right) + \tbinom{1}{1}\right] &= \frac{1}{2}\left[2\left(v - \tbinom{1}{1}\right) + \tbinom{1}{1} - \tbinom{3}{1}\right] + \tbinom{3}{1} \\
&= v + \frac{1}{2}\left[-\tbinom{2}{2} + \tbinom{1}{1} - \tbinom{3}{1}\right] + \tbinom{3}{1} \\
&= v + \frac{1}{2}\left[\tbinom{-4}{-2}\right] + \tbinom{3}{1} \\
&= v - \tbinom{2}{1} + \tbinom{3}{1} \\
&= v + \tbinom{1}{0}
\end{aligned}$$

(Ex. 2) The following puzzle is played on the set of points $\mathbb{Z}^2$ with integer coordinates in $\mathbb{R}^2$. The points $p_1 = (0,0), p_2 = (1,-1)$, and $p_3 = (-2,1)$ are 'mirrors', and the player has a peg placed on some point. A move consists of jumping with the peg across any of the three mirrors. For instance, if the peg is at the point $(1,0)$, we can jump to $(-1,0), (1,-2)$, or $(-5,2)$, depending on the mirror we use. Find a sequence of jumps that takes a peg at position $(1,0)$ to position $(1,2)$ that is different from the solution below.

Another formulation of the problem asks to find a word R in R_1, R_2, R_3, that, when interpreted as a composition of the affine transformations $R_i(v) = -(v - p_i) + p_i$, becomes the translation $R(v) = v + \binom{0}{2}$

Solution: I found two solutions given by (using the notation of words R): $R_1R_3R_1R_2R_1R_2$ and $R_1R_2R_1R_3R_1R_2$. Note that these are different from the giving solution since that solution is given by $R_1R_2R_1R_2R_1R_3$.

To show that these two solutions work, let us write: $R_i = -(v - p_i) + p_i = 2p_i - v$, i.e.:

$$R_1(v) = 2\begin{pmatrix}0\\0\end{pmatrix} - v = -v; \quad R_2(v) = 2\begin{pmatrix}1\\-1\end{pmatrix} - v = \begin{pmatrix}2\\-2\end{pmatrix} - v; \quad R_3(v) = 2\begin{pmatrix}-2\\1\end{pmatrix} - v = \begin{pmatrix}-4\\2\end{pmatrix} - v$$

So that:

i) $(R_1R_3R_1R_2R_1R_2)(v) = (R_1R_3R_1R_2R_1)(\binom{2}{-2} - v) = (R_1R_3R_1R_2)(v - \binom{2}{-2}) = (R_1R_3R_1)((\binom{2}{-2} - v + \binom{2}{-2})) = (R_1R_3R_1)(\binom{4}{-4} - v) = (R_1R_3)(v - \binom{4}{-4}) = R_1(\binom{-4}{2} - v + \binom{4}{-4}) = R_1(\binom{0}{-2} - v) = v - \binom{0}{-2} = v + \binom{0}{2}$

ii) $(R_1R_2R_1R_3R_1R_2)(v) = (R_1R_2R_1R_3R_1)(\binom{2}{-2} - v) = (R_1R_2R_1R_3)(v - \binom{2}{-2}) = (R_1R_2R_1)(\binom{-4}{2} - v + \binom{2}{-2}) = (R_1R_2R_1)(\binom{-2}{0} - v) = (R_1R_2)(v - \binom{-2}{0}) = (R_1)(\binom{2}{-2} - v + \binom{-2}{0}) = (R_1)(\binom{0}{-2} - v) = v - \binom{0}{-2} = v + \binom{0}{2}$

(Ex. 3) Consider the projective plane $\mathbb{F}_3P^2$ over the field with 3 elements. Show that the two triangles with vertices at $p_1 = (1:1:0), p_2 = (1:2:1), p_3 = (0:2:1)$ and $q_1 = (1:0:0)$, $q_2 = (1:1:1), q_3 = (0:0:1)$ are in perspective centrally. Then verify Desargue's theorem by computing the three intersections of corresponding lines (like p_1p_2 with q_1q_2), and showing that they are collinear.

Solution: To show that the two triangles are in perspective centrally, let us compute the intersection of the following lines: p_1q_1 and p_2q_2, p_1q_1 and p_3q_3, p_2q_2 and p_3q_3.

p_1q_1 and p_2q_2 : $p_1 \times q_1 = (0,0,-1) \Longrightarrow -z = 0 \iff z = 0 \Longrightarrow p_1q_1 = \{(x:y:0) \in \mathbb{F}_3P^2\}$
$p_2 \times q_2 = (1,0,-1) \Longrightarrow x - z = 0 \iff x = z \Longrightarrow p_2q_2 = \{(x:y:x) \in \mathbb{F}_3P^2\}$
The intersection is given by $z = 0 = x \Longrightarrow (0:y:0)$, a representative point would be $\boxed{(0:1:0)}$

p_1q_1 and p_3q_3 : We already know that $p_1q_1 = \{(x:y:0) \in \mathbb{F}_3P^2\}$
$p_3 \times q_3 = (2,0,0) \Longrightarrow 2x = 0 \iff x = 0 \Longrightarrow p_3q_3 = \{(0:y:z) \in \mathbb{F}_3P^2\}$
The intersection is given by $z = 0$ and $x = 0 \Longrightarrow (0:y:0)$, a representative point would be $\boxed{(0:1:0)}$

p_2q_2 and p_3q_3 : We already know that $p_2q_2 = \{(x:y:x) \in \mathbb{F}_3P^2\}$
We already know that $p_3q_3 = \{(0:y:z) \in \mathbb{F}_3P^2\}$
The intersection is given by $z = x = 0 \Longrightarrow (0:y:0)$, a representative point would be $\boxed{(0:1:0)}$

Showing that the point $\boxed{(0:1:0)}$ is the center of perspective, i.e., the two triangles are in perspective centrally.

Now, let us verify Desargue's theorem: first find r_{ij} the intersection of p_ip_j and q_iq_j for $i \neq j$

r_{12}: $p_1 \times p_2 = (1,-1,1) \Longrightarrow p_1p_2 = \{(x:y:z) \in \mathbb{F}_3P^2 : x - y + z = 0\}$
$q_1 \times q_2 = (0,-1,1) \Longrightarrow q_1q_2 = \{(x:y:y) \in \mathbb{F}_3P^2\}$
Hence, the intersection is given by $x - y + z = 0$ and $y = z \Longrightarrow x - z + z = 0 \iff x = 0$, so

$$\boxed{r_{12} = (0:1:1)}$$

r_{13}: $p_1 \times p_3 = (1,-1,2) \Longrightarrow p_1p_3 = \{(x:y:z) \in \mathbb{F}_3P^2 : x - y + 2z = 0\}$
$q_1 \times q_3 = (0,-1,0) \Longrightarrow q_1q_3 = \{(x:0:z) \in \mathbb{F}_3P^2\}$
Hence, the intersection is given by $x - y + 2z = 0$ and $y = 0 \Longrightarrow x - 0 + 2z = 0 \iff x = -2z$, so

$$\boxed{r_{13} = (-2:0:1)}$$

r_{23}: $p_2 \times p_3 = (0,-1,2) \Longrightarrow p_2p_3 = \{(x:2z:z) \in \mathbb{F}_3P^2\}$
$q_2 \times q_3 = (1,-1,0) \Longrightarrow q_2q_3 = \{(x:x:z) \in \mathbb{F}_3P^2\}$
Hence, the intersection is given by $y = 2z$ and $x = y \Longrightarrow x = y = 2z$, so

$$\boxed{r_{23} = (2:2:1)}$$

Next, we can find the line through r_{12} and r_{13} by computing $r_{12} \times r_{13} = (1,-2,2)$, so the line is

$$r_{12}r_{13} = \{(x:y:z) \in \mathbb{F}_3P^2 : x - 2y + 2z = 0\}$$

Finally, note that r_{23} is in this line since it satisfies: $2 - 2(2) + 2(1) = 2 - 4 + 2 = 0$, showing that they are collinear.

(Ex. 4) Show that in the projective plane $\mathbb{F}_3P^2$ over the field with 3 elements, the set of points and lines form a configuration of type 13_4.

Solution: First, let us count how many points and lines are in the projective plane $\mathbb{F}_3P^2$. Let $P = \{\text{points in } \mathbb{F}_3P^2\}$. Then, $|P| = (3 \cdot 3 \cdot 3 - 1)/2 = 26/2 = 13$, because there are three choices for the first coordinates, three for the second and three for the third. We discard the point $(0:0:0)$ which is not in $\mathbb{F}_3P^2$. Finally, we divide by 2 because we counted each point exactly twice, the repetition coming from the point being multiplied by 2.

Similarly, let $L = \{\text{lines in} \mathbb{F}_3P^2\}$. Then, $|L| = (3 \cdot 3 \cdot 3 - 1)/2 = 26/2 = 13$. In this case we know that a line is given by $a_1x + a_2y + a_3z = 0$, where $a_1, a_2, a_3 \in \mathbb{F}_3$. Again, there are three choices for a_1, three choices for a_2 and three choices for a_3. Discard the choice $a_1 = a_2 = a_3 = 0$. We over counted each line twice since we line given by $(a_1 : a_2 : a_3)$ is exactly the same as the line given by $(2a_1 : 2a_3 : 2a_3)$.

Let us find each point and each line and show that in each line contains exactly 4 points and that each point is concurrent with exactly 4 lines. $P = \{(0:0:1), (0:1:0), (1:0:0), (0:1:1), (1:0:1), (1:1:0), (0:1:2), (1:0:2), (1:2:0), (1:1:1), (1:1:2), (1:2:1), (2:1:1)\}$ The set of lines can be interpreted as follow: if $(x : y : z) \in P$, then $x + y + z = 0$ defines a line. For the points:

$(0:0:1)$ is in the lines (1) $x = 0$, (2) $y = 0$, (3) $x + y = 0$ and (4) $x + 2y = 0$.

$(0:1:0)$ is in the lines (1) $x = 0$, (2) $z = 0$, (3) $x + z = 0$ and (4) $x + 2z = 0$.

$(1:0:0)$ is in the lines (1) $y = 0$, (2) $z = 0$, (3) $y + z = 0$ and (4) $y + 2z = 0$.

$(0:1:1)$ is in the lines (1) $x = 0$, (2) $y + 2z = 0$, (3) $x + 2y + z = 0$ and (4) $x + y + 2z = 0$.

$(1:0:1)$ is in the lines (1) $y = 0$, (2) $x + 2z = 0$, (3) $x + y + 2z = 0$ and (4) $2x + y + z = 0$.

$(1:1:0)$ is in the lines (1) $z = 0$, (2) $x + 2y = 0$, (3) $x + 2y + z = 0$ and (4) $2x + y + z = 0$.

$(0:1:2)$ is in the lines (1) $x = 0$, (2) $y + z = 0$, (3) $x + y + z = 0$ and (4) $2x + y + z = 0$.

$(1:0:2)$ is in the lines (1) $y = 0$, (2) $x + z = 0$, (3) $x + y + z = 0$ and (4) $x + 2y + z = 0$.

$(1:2:0)$ is in the lines (1) $z = 0$, (2) $x + y = 0$, (3) $x + y + z = 0$ and (4) $x + y + 2z = 0$.

$(1:1:1)$ is in the lines (1) $x + y + z = 0$, (2) $x + 2y = 0$, (3) $x + 2z = 0$ and (4) $y + 2z = 0$.

$(1:1:2)$ is in the lines (1) $x + 2y = 0$, (2) $x + z = 0$, (3) $y + z = 0$ and (4) $x + y + 2z = 0$.

$(1:2:1)$ is in the lines (1) $x + y = 0$, (2) $y + z = 0$, (3) $x + 2z = 0$ and (4) $x + 2y + z = 0$.

$(2:1:1)$ is in the lines (1) $x + y = 0$, (2) $x + z = 0$, (3) $y + 2z = 0$ and (4) $2x + y + z = 0$.

Now, for the lines:

$x = 0$ contains the points: (1) $(0:0:1)$,(2) $(0:1:0)$, (3) $(0:1:1)$ and (4) $(0:1:2)$.

$y = 0$ contains the points: (1) $(0:0:1)$,(2) $(1:0:0)$, (3) $(1:0:1)$ and (4) $(1:0:2)$.

$z = 0$ contains the points: (1) $(0:1:0)$,(2) $(1:0:0)$, (3) $(1:1:0)$ and (4) $(1:2:0)$.

$x + y = 0$ contains the points: (1) $(0:0:1)$, (2) $(1:2:1)$,(3) $(1:2:0)$ and (4) $(2:1:1)$.

$x + z = 0$ contains the points: (1) $(0:1:0)$, (2) $(1:0:2)$, (3) $(1:1:2)$ and (4) $(2:1:1)$.

$y + z = 0$ contains the points: (1) $(1:0:0)$, (2) $(0:1:2)$, (3) $(1:1:2)$ and (4) $(1:2:1)$.

$x + 2y = 0$ contains the points: (1) $(0:0:1)$, (2) $(1:1:0)$, (3) $(1:1:1)$ and (4) $(1:1:2)$.

$x + 2z = 0$ contains the points: (1) $(0:1:0)$, (2) $(1:0:1)$, (3) $(1:2:1)$ and (4) $(1:1:1)$.

$y + 2z = 0$ contains the points: (1) $(1:0:0)$, (2) $(0:1:1)$, (3) $(1:1:1)$ and (4) $(2:1:1)$.

$x + y + z = 0$ contains the points: (1) $(0:1:2)$, (2) $(1:0:2)$, (3) $(1:2:0)$ and (4) $(1:1:1)$.

$x + y + 2z = 0$ contains the points: (1) $(0:1:1)$, (2) $(1:0:1)$, (3) $(1:2:0)$ and (4) $(1:1:2)$.

$x + 2y + z = 0$ contains the points: (1) $(0:1:1)$, (2) $(1:1:0)$, (3) $(1:0:2)$ and (4) $(1:2:1)$.

$2x + y + z = 0$ contains the points: (1) $(1:0:1)$, (2) $(1:1:0)$, (3) $(0:1:2)$ and (4) $(2:1:1)$.

(Ex. 5) Show that the Hesse configuration can be realized in the complex projective plane $\mathbb{C}P^2$ by writing

$$\begin{array}{lll} p_{00} = (0:-1:1) & p_{01} = (-1:0:1) & p_{02} = (-1:1:0) \\ p_{10} = (0:y:1) & p_{11} = (x:0:1) & p_{12} = (y:1:0) \\ p_{20} = (0:x:1) & p_{21} = (y:0:1) & p_{22} = (x:1:0) \end{array}$$

for suitable complex numbers $x \neq y$

Solution: If the Hesse configuration is to be realized in the complex projective plane, then the points p_{10}, p_{11} and p_{12} need to be collinear. Using the determinant condition for collinearity of points we get that:

$$det\left[\begin{pmatrix} 0 & y & 1 \\ x & 0 & 1 \\ y & 1 & 0 \end{pmatrix}\right] = 0 \iff -y\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix} + \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = 0 \iff -y(-y) + x = 0 \iff y^2 + x = 0$$

Likewise, the points p_{20}, p_{21} and p_{22} need to be collinear and so:

$$det\left[\begin{pmatrix} 0 & x & 1 \\ y & 0 & 1 \\ x & 1 & 0 \end{pmatrix}\right] = 0 \iff -x\begin{pmatrix} y & 1 \\ x & 0 \end{pmatrix} + \begin{pmatrix} y & 0 \\ x & 1 \end{pmatrix} = 0 \iff -x(-x) + y = 0 \iff x^2 + y = 0$$

Also, the points p_{10}, p_{01} and p_{22} need to be collinear:

$$det\left[\begin{pmatrix} 0 & y & 1 \\ -1 & 0 & 1 \\ x & 1 & 0 \end{pmatrix}\right] = 0 \iff -y\begin{pmatrix} -1 & 1 \\ x & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ x & 1 \end{pmatrix} = 0 \iff -y(-x) - 1 = 0 \iff xy - 1 = 0$$

Again, the points p_{20}, p_{01} and p_{12} need to be collinear:

$$det\left[\begin{pmatrix} 0 & x & 1 \\ -1 & 0 & 1 \\ y & 1 & 0 \end{pmatrix}\right] = 0 \iff -x\begin{pmatrix} -1 & 1 \\ y & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ y & 1 \end{pmatrix} = 0 \iff -x(-y) - 1 = 0 \iff xy - 1 = 0 \text{ this implies } x \neq 0, y \neq 0$$

Note that any other choice of three points p_{ij} satisfying the Hesse configuration, i.e., being collinear in the Hesse configuration, will yield a zero determinant, providing no further information. Therefore, we have the following system:

$$\left\{\begin{array}{llll} y^2 + x = 0 & \implies (-x^2)^2 + x = 0 & \implies x^4 + x = 0 & \implies x(x^3+1) = 0 \\ x^2 + y = 0 & \quad\; y = -x^2 & & \\ xy - 1 = 0 & & & \end{array}\right\}$$

We need to find the solutions of $x(x^3 + 1) = 0$ and then solve for y. The equation $x(x^3 + 1) = 0$ implies that $x = 0$ OR $x^3 - 1 = 0$. The solution $x = 0$ contradicts the equation $xy - 1 = 0$, so we discard this solution. The problem reduces to finding all roots of the polynomial $x^3 + 1$. Clearly, one root is -1 since, $-1^3 + 1 = -1 + 1 = 0$. Hence, the polynomial $x^3 + 1$ is divisible by $(x + 1)$, yielding: $x^3 + 1 = (x + 1)(x^2 - x + 1)$.

Applying quadratic formula: $x^2 - x + 1 = 0 \iff x = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \sqrt{3}i}{2}$. Therefore,

$$x^3 + 1 = (x + 1)(x^2 - x + 1) = (x + 1)(x - \left(\frac{1+\sqrt{3}i}{2}\right))(x - \left(\frac{1-\sqrt{3}i}{2}\right))$$

Finally, we can solve for y :

i) If $x = -1$ then $(-1)y - 1 = 0 \iff y = -1$. So $x = y = -1$. But we discard this solution since we need $x \neq y$.

ii) If $x = \left(\dfrac{1+\sqrt{3}i}{2}\right)$ then $\left(\dfrac{1+\sqrt{3}i}{2}\right) y - 1 = 0 \iff y = \left(\dfrac{2}{1+\sqrt{3}i}\right) = \left(\dfrac{2}{1+\sqrt{3}i}\right)\left(\dfrac{1-\sqrt{3}i}{1-\sqrt{3}i}\right) = \left(\dfrac{1-\sqrt{3}i}{2}\right)$

Therefore, one complex solution is $\boxed{(x, y) = \left(\dfrac{1+\sqrt{3}i}{2}, \dfrac{1-\sqrt{3}i}{2}\right)}$

iii) If $x = \left(\dfrac{1-\sqrt{3}i}{2}\right)$ then $\left(\dfrac{1-\sqrt{3}i}{2}\right) y - 1 = 0 \iff y = \left(\dfrac{2}{1-\sqrt{3}i}\right) = \left(\dfrac{2}{1-\sqrt{3}i}\right)\left(\dfrac{1+\sqrt{3}i}{1+\sqrt{3}i}\right) = \left(\dfrac{1+\sqrt{3}i}{2}\right)$

Finally, another complex solution is $\boxed{(x, y) = \left(\dfrac{1-\sqrt{3}i}{2}, \dfrac{1+\sqrt{3}i}{2}\right)}$