M436 - Introduction to Geometries - Homework 2

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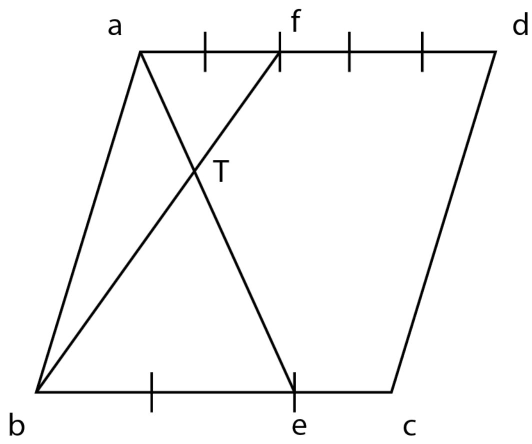
(Ex. 1) **Solution:** In the cover page.

(Ex. 2) Is the configuration in **Figure 2** isomorphic to the Pappus configuration? Justify your decision.

Solution: No, the configuration in **Figure 2** is not isomorphic to the Pappus configuration. Here is why: Choose an arbitrary point in the Pappus configuration. This point is not collinear with exactly two other points. Moreover, these two points are not collinear among themselves. Now, choose an arbitrary point in the configuration in **Figure 2**. Again, this point is not collinear with exactly two other points. However, these two points are collinear among themselves. This shows that the two configurations are not isomorphic.

(Ex. 3) In a parallelogram $abcd$ the edge bc is divided by e in the proportion $2:1$, and the edge ad is divided by f in the proportion $2:3$. Determine in what proportions the segments ae and bf intersect each other.

Solution: Consider the following arbitrary parallelogram:



Let v be the vector $\overrightarrow{bc}$ and w be the vector $\overrightarrow{ba}$. The point of intersection T can be written as:

$$T = (1-s)b + sf$$

$$T = (1-t)e + ta$$

Now, we can write all these points as linear combinations of v and w as follow: first note that $b = 0v + 0w = 0$ (the "origin" of our parallelogram). Next, $f = \frac{2}{5}v + w$, since the edge ad is divided by f in the proportion $2:3$. Likewise, $e = \frac{2}{3}v$, since the edge bc is divided by e in the proportion $2:1$. Finally, $a = w$. Replacing in previous equations:

$$\begin{array}{l} T = (1-s)0 + s\left(\frac{2}{5}v + w\right) = \frac{2}{5}sv + sw \\ T = (1-t)\frac{2}{3}v + tw = \frac{2}{3}v - \frac{2}{3}tv + tw \end{array} \implies \text{(equating the two equations)} \quad \frac{2}{5}sv + sw = \frac{2}{3}v - \frac{2}{3}tv + tw \implies$$

$$\frac{2}{5}sv + sw - \frac{2}{3}v + \frac{2}{3}tv - tw = 0 \implies v\left(\frac{2}{5}s - \frac{2}{3} + \frac{2}{3}t\right) + w(s-t) = 0 \implies \text{but } v \text{ and } w \text{ are linearly independent} \implies$$

$$\begin{array}{l} \frac{2}{5}s - \frac{2}{3} + \frac{2}{3}t = 0 \\ s - t = 0, \text{ so } s = t \end{array} \implies \frac{2}{5}s - \frac{2}{3} + \frac{2}{3}s = 0 \implies \frac{16}{15}s = \frac{2}{3} \implies s = \frac{5}{8}$$

Hence, $s = t = \dfrac{5}{8}$. Therefore, the proportion in which the segments ae and bf intersect each other is

$$\sigma = \frac{s}{1-s} = \frac{\frac{5}{8}}{1-\frac{5}{8}} = \frac{\frac{5}{8}}{\frac{3}{8}} = \frac{5}{3}$$

That is, $\boxed{\tau = \sigma = 5:3}$

(Ex. 4) Consider the triangle $\Delta(p_1p_2p_3)$ with

$$p_1 = (1,-1) \qquad p_2 = (4,1) \qquad p_3 = (2,3)$$

and the line through $(4,-9)$ and $(1,19)$. Compute the ratios r_i with which this line intersects the three lines p_1p_2, p_2p_3 and p_3p_1. Then compute $r_1r_2r_3$.

Solution:

For r_1 : First, let us find the point of intersection between p_1p_2 and the line l, call this point T.

$$\vec{l}(s) := \begin{pmatrix}1\\19\end{pmatrix} + s\left[\begin{pmatrix}4\\-9\end{pmatrix} - \begin{pmatrix}1\\19\end{pmatrix}\right] = \begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_1p_2}(t) := \begin{pmatrix}1\\-1\end{pmatrix} + t\left[\begin{pmatrix}4\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}\right] = \begin{pmatrix}1\\-1\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\vec{l} = \overrightarrow{p_1p_2}$, which means:

$$\begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{matrix}1+3s = 1+3t \\ 19-28s = -1+2t\end{matrix} \implies \begin{matrix}s = t \\ 19+1 = 28s+2t\end{matrix} \implies 20 = 28t + 2t \implies t = 2/3$$

Therefore, $t = \dfrac{2}{3}$ is the parameter for line $\overrightarrow{p_1p_2}$ for which it intersects with $\vec{l}$. The coordinates of the intersection are:

$$\overrightarrow{p_1p_2}\,(2/3) = \begin{pmatrix}1\\-1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}2\\4/3\end{pmatrix} = \begin{pmatrix}3\\1/3\end{pmatrix}$$

So line $\vec{l}$ intersects line $\overrightarrow{p_1p_2}$ at the point $T = (3, 1/3)$. Now we can find the ratio r_1:

$$\begin{pmatrix}3\\1/3\end{pmatrix} = T = (1-t)\begin{pmatrix}1\\-1\end{pmatrix} + t\begin{pmatrix}4\\1\end{pmatrix} \implies 3 = 1 - t + 4t = 1 + 3t \implies t = 2/3$$

Finally, the proportion is given by $\tau_1 = \dfrac{2/3}{1-2/3} = 2$, so that $\boxed{r_1 = \dfrac{2}{1}}$

For r_2 : First, let us find the point of intersection between p_2p_3 and the line l, call this point T.

$$\vec{l}(s) := \begin{pmatrix}1\\19\end{pmatrix} + s\left[\begin{pmatrix}4\\-9\end{pmatrix} - \begin{pmatrix}1\\19\end{pmatrix}\right] = \begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_2p_3}(t) := \begin{pmatrix}4\\1\end{pmatrix} + t\left[\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}4\\1\end{pmatrix}\right] = \begin{pmatrix}4\\1\end{pmatrix} + t\begin{pmatrix}-2\\2\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\vec{l} = \overrightarrow{p_2p_3}$, which means:

$$\begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} + t\begin{pmatrix}-2\\2\end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{matrix}1+3s = 4-2t \\ 19-28s = 1+2t\end{matrix} \implies \text{(adding the two equations)} \quad 20 - 25s = 5 \implies s = 3/5$$

Therefore, $s = \dfrac{3}{5}$ is the parameter for line $\vec{l}$ for which it intersects with $\overrightarrow{p_2p_3}$. The coordinates of the intersection are:

$$\vec{l}\,(3/5) = \begin{pmatrix}1\\19\end{pmatrix} + \frac{3}{5}\begin{pmatrix}3\\-28\end{pmatrix} = \begin{pmatrix}1\\19\end{pmatrix} + \begin{pmatrix}9/5\\-84/5\end{pmatrix} = \begin{pmatrix}14/5\\11/5\end{pmatrix}$$

So line $\vec{l}$ intersects line $\overrightarrow{p_2p_3}$ at the point $T = (14/5, 11/5)$. Now we can find the ratio r_2:

$$\begin{pmatrix}14/5\\11/5\end{pmatrix} = T = (1-t)\begin{pmatrix}4\\1\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix} \implies 11/5 = 1 - t + 3t = 1 + 2t \implies 2t = 6/5 \implies t = 3/5$$

Finally, the proportion is given by $\tau_2 = \dfrac{3/5}{1-3/5} = 3/2$, so that $\boxed{r_2 = \dfrac{3}{2}}$

For r_3 : First, let us find the point of intersection between p_3p_1 and the line l, call this point T.

$$\overrightarrow{l}(s) := \begin{pmatrix}1\\19\end{pmatrix} + s\left[\begin{pmatrix}4\\-9\end{pmatrix} - \begin{pmatrix}1\\19\end{pmatrix}\right] = \begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_3p_1}(t) := \begin{pmatrix}2\\3\end{pmatrix} + t\left[\begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}\right] = \begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}-1\\-4\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\overrightarrow{l} = \overrightarrow{p_3p_1}$, which means:

$$\begin{pmatrix}1\\19\end{pmatrix} + s\begin{pmatrix}3\\-28\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}-1\\-4\end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{array}{lll} 1+3s = 2-t & \implies & 1-3s = t \\ 19-28s = 3-4t & & 19-28s = 3-4[1-3s] \implies 19-28s = -1+12s \implies s = 1/2 \end{array}$$

Therefore, $s = \dfrac{1}{2}$ is the parameter for line $\overrightarrow{l}$ for which it intersects with $\overrightarrow{p_3p_1}$. The coordinates of the intersection are:

$$\overrightarrow{l}(1/2) = \begin{pmatrix}1\\19\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\-28\end{pmatrix} = \begin{pmatrix}1\\19\end{pmatrix} + \begin{pmatrix}3/2\\-14\end{pmatrix} = \begin{pmatrix}5/2\\5\end{pmatrix}$$

So line $\overrightarrow{l}$ intersects line $\overrightarrow{p_3p_1}$ at the point $T = (5/2, 5)$. Now we can find the ratio r_3:

$$\begin{pmatrix}5/2\\5\end{pmatrix} = T = (1-t)\begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix} \implies 5 = 3-3t-t \implies 2 = -4t \implies t = -1/2$$

Finally, the proportion is given by $r_3 = \dfrac{-1/2}{1-(-1/2)} = -1/3$, so that $\boxed{r_3 = -\dfrac{1}{3}}$

Therefore, $\boxed{r_1r_2r_3 = \frac{2}{1}\frac{3}{2}(-\frac{1}{3}) = -1}$

(Ex. 5) Let six points on the unit circle be given as $p_1 = (-4/5, -3/5), p_2 = (5/13, -12/13), p_3 = (4/5, -3/5)$ and $q_1 = (-3/5, 4/5), q_2 = (0, 1), q_3 = (4/5, 3/5)$. Denote the intersection of the two lines p_iq_j and p_jq_i by r_{ij} for $i \neq j$. Compute the coordinates of r_{12}, r_{13} and r_{23}, and show that the three points are collinear.

Solution:

For r_{12}: We need the intersection of p_1q_2 and p_2q_1:

$$\overrightarrow{p_1q_2}(s) := \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + s\left[\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-4/5\\-3/5\end{pmatrix}\right] = \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + s\begin{pmatrix}4/5\\8/5\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_2q_1}(t) := \begin{pmatrix}5/13\\-12/13\end{pmatrix} + t\left[\begin{pmatrix}-3/5\\4/5\end{pmatrix} - \begin{pmatrix}5/13\\-12/13\end{pmatrix}\right] = \begin{pmatrix}5/13\\-12/13\end{pmatrix} + t\begin{pmatrix}-64/65\\112/65\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1q_2} = \overrightarrow{p_2q_1}$

$$\left(\begin{smallmatrix}-4/5\\-3/5\end{smallmatrix}\right) + s\left(\begin{smallmatrix}4/5\\8/5\end{smallmatrix}\right) = \left(\begin{smallmatrix}5/13\\-12/13\end{smallmatrix}\right) + t\left(\begin{smallmatrix}-64/65\\112/65\end{smallmatrix}\right) \implies \begin{array}{l} -4/5+4/5s = 5/13-64/65t \\ -3/5+8/5s = -12/13+112/65t \end{array} \implies 4/5s = 5/13+4/5-64/65t \implies$$

$$4/5s = 77/65 - 64/65t \implies 8/5s = 2[77/65-64/65t] \implies \text{(replace in 2n eq.)}$$

$$-3/5 + 2[77/65 - 64/65t] = -12/13 + 112/65t \implies 35/13 = 240/65t \implies t = 35/48$$

The coordinates of r_{12} are given by

$$\overrightarrow{p_2q_1}(35/48) = \begin{pmatrix}5/13\\-12/13\end{pmatrix} + \frac{35}{48}\begin{pmatrix}-64/65\\112/65\end{pmatrix} = \begin{pmatrix}-1/3\\1/3\end{pmatrix}$$

Hence,

$$r_{12} = \boxed{(-1/3, 1/3)}$$

For r_{13}: We need the intersection of p_1q_3 and p_3q_1:

$$\overrightarrow{p_1q_3}(s) := \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + s\left[\begin{pmatrix}4/5\\3/5\end{pmatrix} - \begin{pmatrix}-4/5\\-3/5\end{pmatrix}\right] = \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + s\begin{pmatrix}8/5\\6/5\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_3q_1}(t) := \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\left[\begin{pmatrix}-3/5\\4/5\end{pmatrix} - \begin{pmatrix}4/5\\-3/5\end{pmatrix}\right] = \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\begin{pmatrix}-7/5\\7/5\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1q_3} = \overrightarrow{p_3q_1}$

$$\begin{pmatrix}-4/5\\-3/5\end{pmatrix} + s\begin{pmatrix}8/5\\6/5\end{pmatrix} = \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\begin{pmatrix}-7/5\\7/5\end{pmatrix} \implies \begin{array}{l}-4/5 + 8/5s = 4/5 - 7/5t\\ -3/5 + 6/5s = -3/5 + 7/5t\end{array} \implies \text{add the two equations} \implies$$

$$-3/5 + 6/5s - 4/5 + 8/5s = -3/5 + 4/5 \implies 14/5s = 8/5 \implies s = 4/7$$

The coordinates of r_{13} are given by

$$\overrightarrow{p_1q_3}\,(4/7) = \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + \frac{4}{7}\begin{pmatrix}8/5\\6/5\end{pmatrix} = \begin{pmatrix}-4/5\\-3/5\end{pmatrix} + \begin{pmatrix}32/35\\24/35\end{pmatrix} = \begin{pmatrix}4/35\\3/35\end{pmatrix}$$

Hence,

$$r_{13} = \boxed{(4/35, 3/35)}$$

For r_{23}: We need the intersection of p_2q_3 and p_3q_2:

$$\overrightarrow{p_2q_3}(s) := \begin{pmatrix}5/13\\-12/13\end{pmatrix} + s\left[\begin{pmatrix}4/5\\3/5\end{pmatrix} - \begin{pmatrix}5/13\\-12/13\end{pmatrix}\right] = \begin{pmatrix}5/13\\-12/13\end{pmatrix} + s\begin{pmatrix}27/65\\99/65\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_3q_2}(t) := \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\left[\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}4/5\\-3/5\end{pmatrix}\right] = \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\begin{pmatrix}-4/5\\8/5\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_2q_3} = \overrightarrow{p_3q_2}$

$$\begin{pmatrix}5/13\\-12/13\end{pmatrix} + s\begin{pmatrix}27/65\\99/65\end{pmatrix} = \begin{pmatrix}4/5\\-3/5\end{pmatrix} + t\begin{pmatrix}-4/5\\8/5\end{pmatrix} \implies \begin{array}{l}5/13 + 27/65s = 4/5 - 4/5t\\ -12/13 + 99/65s = -3/5 + 8/5t\end{array} \implies \begin{array}{l}\text{multiply 1st eq. by 2 and}\\ \text{add to 2nd eq.}\end{array} \implies$$

$$-12/13 + 99/65s + 10/13 + 54/65s = -3/5 + 8/5 \implies 153/65s - 2/13 = 1 \implies s = 25/51$$

The coordinates of r_{23} are given by

$$\overrightarrow{p_2q_3}\,(25/51) = \begin{pmatrix}5/13\\-12/13\end{pmatrix} + \frac{25}{51}\begin{pmatrix}27/65\\99/65\end{pmatrix} = \begin{pmatrix}5/13\\-12/13\end{pmatrix} + \begin{pmatrix}45/221\\165/221\end{pmatrix} = \begin{pmatrix}10/17\\-3/17\end{pmatrix}$$

Hence,

$$r_{23} = \boxed{(10/17, -3/17)}$$

Now, the equation for the line through r_{12} and r_{13} is given by:

$$\overrightarrow{r_{12}r_{13}}(s) := \begin{pmatrix}-1/3\\1/3\end{pmatrix} + s\left[\begin{pmatrix}4/35\\3/35\end{pmatrix} - \begin{pmatrix}-1/3\\1/3\end{pmatrix}\right] = \begin{pmatrix}-1/3\\1/3\end{pmatrix} + s\begin{pmatrix}47/105\\-26/105\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

Solving this vector equation in terms of x and y we get:

$$\begin{pmatrix}-1/3\\1/3\end{pmatrix} + s\begin{pmatrix}47/105\\-26/105\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \implies \begin{array}{l}1/3 + 47/105s = x\\ 1/3 - 26/105s = y\end{array} \implies (x + 1/3)(105/47) = s \implies \text{Replace in 2nd eq.}$$

$$y = 1/3 - (26/105)[(x + 1/3)(105/47)] \implies y = 1/3 - (26/47)(x + 1/3) \implies y = 7/47 - 26/47x$$

Finally, note that point r_{23} satisfies this equation since $-\frac{3}{17} = \frac{7}{47} - \frac{26}{47}\frac{10}{17}$. Therefore, all three points r_{12}, r_{13} and r_{23} lie in the line $y = 7/47 - 26/47x$, meaning that they are collinear.