M/S413 Final Exam

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All problems from Rudin, chapter 5 on differentiation pages 114-119.

(4) If
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$
where $C_0, \cdots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

Proof: Let $f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$ Note that:

$$\begin{array}{lclcl cl}
f(0) & = & C_0 0 + \frac{C_1}{2}0^2 + \cdots + \frac{C_{n-1}}{n}0^n + \frac{C_n}{n+1}0^{n+1} & = & 0 + 0 + \cdots + 0 + 0 & = & 0 \\
f(1) & = & C_0 1 + \frac{C_1}{2}1^2 + \cdots + \frac{C_{n-1}}{n}1^n + \frac{C_n}{n+1}1^{n+1} & = & C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} & = & 0 \textit{ (by hypothesis)}
\end{array}$$

Hence $f(0) = f(1) = 0$.

Now, let us apply the Mean Value Theorem by first checking that f satisfies its hypothesis on $(0, 1)$:

(i) The function $f : [0, 1] \to \mathbb{R}$ is continuos on $[0, 1]$. This is because f is a polynomial which we know is continuos on its domain.

(ii) The function f is differentiable on $(0, 1)$. Again, this is because f is a polynomial which we know is differentiable on its domain.

$\therefore$ There exists $x_0 \in (0, 1)$ such that:
$$\frac{f(1) - f(0)}{1 - 0} = f'(x_0) \iff f'(x_0) = f(1) - f(0) \iff f'(x_0) = 0 - 0 \iff f'(x_0) = 0$$

But note that $f'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$, and by the above result we have found $x_0 \in (0, 1)$ s.t:
$$C_0 + C_1 x_0 + \cdots + C_{n-1}x_0^{n-1} + C_n x_0^n = 0$$
So x_0 is one real root in $(0, 1)$. Hence, we can conclude that the given equation has at least one real root.

(5) Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$.
Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$

Proof: Let $x \in \mathbb{R}$, $x > 0$. Let us apply the Mean Value Theorem by first checking that f satisfies its hypothesis on $(x, x + 1)$:

(i) The function $f : [x, x+1] \to \mathbb{R}$ is continuos on $[x, x+1]$. This is because by hypothesis f is differentiable for every $x > 0$ and by theorem 5.2 we know that a differentiable function on x must be continuos at x.

(ii) The function f is differentiable on $(x, x + 1)$. By hypothesis.

$\therefore$ For all $x > 0$, there exists $y = y(x) \in (x, x + 1)$ such that:
$$\frac{f(x+1) - f(x)}{x + 1 - x} = f'(y) \iff f'(y) = f(x + 1) - f(x) = g(x) \quad \text{By hypothesis}$$

So we can conclude that $f'(y) = g(x)$. Now, note that $y \in (x, x + 1)$ so that $y > x$. Therefore
$$\lim_{x\to\infty} y(x) = \infty \quad \text{(if } x \text{ goes to infinity } y \text{ must go to infinity because } y > x)$$

But then,
$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f'(y) = 0 \quad \text{Since by hypothesis } f'(x) \to 0 \text{ as } x \to +\infty$$

(17) Suppose f is a real, three times differentiable function on $[-1,1]$, such that

$$f(-1)=0, \quad f(0)=0, \quad f(1)=1, \quad f'(0)=0$$

Prove that $f^{(3)}(x)\geq 3$ for some $x\in(-1,1)$.

Note that equality holds for $\frac{1}{2}(x^3+x^2)$.

Hint: Use Theorem 5.15, with $\alpha=0$ and $\beta=\pm 1$, to show that there exist $s\in(0,1)$ and $t\in(-1,0)$ s.t.:

$$f^{(3)}(s)+f^{(3)}(t)=6$$

Proof: Following the hint, let us use theorem 5.15 with $\alpha=0$ and $\beta=\pm 1$.

(i) For $\alpha=0$ and $\beta=1$, there exists $s\in(0,1)$ such that:

$$f(1)=P(1)+\frac{f^{(3)}(s)}{3!}(1-0)^3, \quad \text{where: } P(1)=\frac{f^{(0)}(0)}{0!}(1-0)^0+\frac{f^{(1)}(0)}{1!}(1-0)^1+\frac{f^{(2)}(0)}{2!}(1-0)^2$$

Hence,

$$f(1)=\frac{f^{(0)}(0)}{0!}(1-0)^0+\frac{f^{(1)}(0)}{1!}(1-0)^1+\frac{f^{(2)}(0)}{2!}(1-0)^2+\frac{f^{(3)}(s)}{3!}(1-0)^3$$

Equivalently,

$$f(1)=f(0)+f^{(1)}(0)+\frac{f^{(2)}(0)}{2}+\frac{f^{(3)}(s)}{6} \quad \cdots\cdots\cdots \ (*)$$

(ii) For $\alpha=0$ and $\beta=-1$, there exists $t\in(0,1)$ such that:

$$f(-1)=P(-1)+\frac{f^{(3)}(t)}{3!}(-1-0)^3, \quad \text{where: } P(-1)=\frac{f^{(0)}(0)}{0!}(-1-0)^0+\frac{f^{(1)}(0)}{1!}(-1-0)^1+\frac{f^{(2)}(0)}{2!}(-1-0)^2$$

Hence,

$$f(-1)=\frac{f^{(0)}(0)}{0!}(-1-0)^0+\frac{f^{(1)}(0)}{1!}(-1-0)^1+\frac{f^{(2)}(0)}{2!}(-1-0)^2+\frac{f^{(3)}(t)}{3!}(-1-0)^3$$

Equivalently,

$$f(-1)=f(0)-f^{(1)}(0)+\frac{f^{(2)}(0)}{2}-\frac{f^{(3)}(t)}{6} \quad \cdots\cdots\cdots \ (\triangle)$$

Now consider $(*)-(\triangle)$:

$$\begin{array}{rcl}
f(1)-f(-1) & = & f(0)+f^{(1)}(0)+\frac{f^{(2)}(0)}{2}+\frac{f^{(3)}(s)}{6}-\left(f(0)-f^{(1)}(0)+\frac{f^{(2)}(0)}{2}-\frac{f^{(3)}(t)}{6}\right) \iff \\
1-0 & = & 2f^{(1)}(0)+\frac{f^{(3)}(s)}{6}+\frac{f^{(3)}(t)}{6} \qquad \text{because } f(0)=f(-1)=0 \text{ and } f(1)=1 \text{ by hypothesis} \iff \\
1 & = & \frac{f^{(3)}(s)}{6}+\frac{f^{(3)}(t)}{6} \qquad \text{because } f^{(1)}(0)=0 \text{ by hypothesis} \iff \\
6 & = & f^{(3)}(s)+f^{(3)}(t) \qquad \text{multiplying by 6}
\end{array}$$

Hence, there exists $s\in(0,1)$ and $t\in(-1,0)$ such that $f^{(3)}(s)+f^{(3)}(t)=6$.

Finally, we know that either s or t or both are going to satisfy the result we want, for suppose (for a contradiction) that both $f^{(3)}(s)<3$ and $f^{(3)}(t)<3$. Then $f^{(3)}(s)+f^{(3)}(t)<3+3=6$, a contradiction with $f^{(3)}(s)+f^{(3)}(t)=6$. Hence, $f^{(3)}(s)\geq 3$ or $f^{(3)}(t)\geq 3$, so $f^{(3)}(x)\geq 3$ for some $x\in(-1,1)$.