M413 - Fall 2013 - HW5 - Enrique Areyan

(9) Let E° denote the set of all interior points of a set E.

30/30

(a) Prove that E° is always open.

Pf: Let us prove that $(E^\circ)^c$ is closed.

Let x be a limit point of $(E^\circ)^c$, and $r>0$. By definition, $N_r(x)\setminus\{x\}\cap(E^\circ)^c \neq \emptyset$. In particular, pick y such that $d(x,y)<\frac{r}{2}$. then $y\in N_r(x)\setminus\{x\}\cap(E^\circ)^c$, so that $y\in N_r(x)$, $y\neq x$, $y\in(E^\circ)^c$. Since $y\in(E^\circ)^c \Rightarrow y\notin E^\circ$ and so y is not an interior point of E. By definition, $\forall r_1>0: N_{r_1}(y)\not\subset E$.

Let $0<r_1<\frac{r}{2}$. Pick $t\in N_{r_1}(y)$ such that $t\notin E$. then,

$$t\in N_{r_1}(y) \Rightarrow d(t,y)<r_1<\frac{r}{2} \Rightarrow d(t,y)<\frac{r}{2}$$

By triangular inequality:

$$d(t,x)\leq d(t,y)+d(x,y)<\frac{r}{2}+\frac{r}{2}=r \Rightarrow d(t,x)<r,$$

So $t\in N_r(x)$ but $t\notin E$. therefore, given $r>0$; $N_r(x)\not\subset E$, which means that x is not an interior point of E $\Leftrightarrow x\notin E^\circ \Leftrightarrow x\in(E^\circ)^c$. thus, $(E^\circ)^c$ contains all of its limit points $\Rightarrow (E^\circ)^c$ is closed $\Rightarrow E^\circ$ is open.

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(b) Prove that E is open if and only if $E^\circ=E$

Pf: ($\Rightarrow$) Suppose that E is open. Want to prove $E^\circ \stackrel{?}{=} E$.

($\subseteq$) Let $x\in E^\circ$. By definition of E°, x is an interior point of E. So, there exists $r>0$, such that $N_r(x)\subset E$. Pick such an r. then, $x\in N_r(x)$, since $d(x,x)=0<r$. therefore, $x\in N_r(x)\subset E \Rightarrow x\in E$.

($\supseteq$) Let $x\in E$. Since E is open, all of its points are interior. In particular x is interior to E, which by definition means that $x\in E^\circ$.

($\subseteq$) and ($\supseteq$) means that $E^\circ=E$.

($\Leftarrow$) Suppose that $E^\circ=E$. Want to prove that E is open.

Let $x\in E$. By hypothesis $x\in E\Rightarrow x\in E^\circ$. So we know that $x\in E^\circ$. therefore x is interior to E. So any point $x\in E$ is interior to E, which by definition means that E is open.

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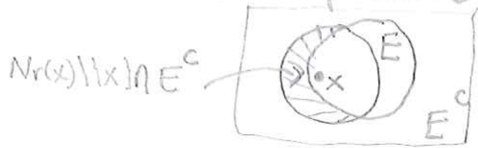
e) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

f: Suppose that $G \subset E$ and G is open.
et $x \in G$. Since G is open, there exists $r > 0$ such that $N_r(x) \subset G$, but $G \subset E$,
$N_r(x) \subset G \subset E \Rightarrow N_r(x) \subset E$; so there exists $r' > 0$ ($r' = r$, pick r that
orks for G), such that $N_{r'}(x) \subset E$, which means that x is interior to E.
t $x \in G$, interior to E, i.e., $x \in E^\circ \Rightarrow G \subset E^\circ$.

d) Prove that the complement of E° is the closure of the complement of E.
: we want to prove $(E^\circ)^c \overset{?}{=} \overline{(E^c)} \Leftrightarrow (E^\circ)^c \overset{?}{=} (E^c \cup (E^c)')$.
=) Let $x \in (E^\circ)^c$. then, $x \notin E^\circ$, by definition means that x is NOT an interior point of E
If $x \notin E$ then $x \in E^c$, and so $x \in E^c \cup (E^c)' \Leftrightarrow \boxed{x \in \overline{E^c}}$.
herwise, If $x \in E$. Let $r > 0$ so that $N_r(x) \not\subset E$. Since $x \in E$, we know that
$_r(x) \setminus \{x\} \not\subset E$. therefore, $N_r(x) \setminus \{x\} \cap E^c \neq \emptyset$. Since this is true for any $r > 0$,
follows that x is a limit point of E^c. So, $x \in E^{c\prime} \Rightarrow x \in E^c \cup E^{c\prime} \Rightarrow \boxed{x \in \overline{E^c}}$.
is final statement in pictures looks like:



) Let $x \in \overline{E^c}$. By definition $x \in E^c \cup E^{c\prime}$.
$x \in E^c$ then $x \notin E$. Let $r > 0$. Consider $N_r(x)$. In particular $x \in N_r(x)$ since
$(x,x) = 0 < r$. therefore, $N_r(x) \not\subset E$, so x is not interior to E, $\Leftrightarrow x \notin E^\circ \Leftrightarrow \boxed{x \in (E^\circ)^c}$
erwise, if $x \in E^{c\prime}$, then x is a limit point of E^c. Let $r > 0$. then,
$_r(x) \setminus \{x\} \cap E^c \neq \emptyset$. Using the same arguments and picture as in (⊆), we can
clude that $N_r(x) \not\subset E$. Hence, x is not interior $E \Leftrightarrow x \notin E^\circ \Leftrightarrow \boxed{x \in (E^\circ)^c}$.
) and (2) means that $(E^\circ)^c = \overline{(E^c)}$

) Do E and $\overline{E}$ always have the same interiors?
ution: NO. Consider $E = (-1,0) \cup (0,1)$. then, by (b) $E^\circ = E$, since E is open.
t, $\overline{E} = [-1,1]$, since $\{0,1,-1\} = E'$, however $(\overline{E})^\circ = (-1,1) \neq (-1,0) \cup (0,1) = E^\circ$.

Do E and E° always have the same closures?
ution: NO. Consider $E = \{0,1\}$. then $\overline{E} = E$, since E is closed (theorem 2.27 (b))
$E^\circ = \emptyset$ (there are no interior points to E). Moreover, $\overline{E^\circ} = E^\circ \cup E^{\circ\prime}$, but
$\emptyset$ (E° has no limit points). thus $\overline{E^\circ} = \emptyset \cup \emptyset = \emptyset \neq \{0,1\} = \overline{E}$.

M413: Fall 2013 - HW5 - Enrique Areyan

(13) Construct a compact set of real numbers whose limit points form a countable set.

Solution: Consider the set:

$$E = \{\frac{1}{p} + \frac{1}{q} : p, q \in \mathbb{N}\} \cup \{0\}.$$

(a) claim: $E' = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

Pf: Clearly, 0 is a limit point of E, since we can make p and q as large as we want so that $\frac{1}{p} + \frac{1}{q} \to 0$, as $p \to \infty$ and $q \to \infty$.

Moreover, if you fix p and let $q \to \infty$, then $\frac{1}{p} + \frac{1}{q} \to \frac{1}{p}$ as $q \to \infty$. this shows that we can get as close as we want to $\frac{1}{p}$; $p \in \mathbb{N}$, with numbers from E. More concretely, Let $\varepsilon > 0$, and $p \in \mathbb{N}$. Pick $q \in \mathbb{N}$, such that $q \neq p$, $q >$ (we can do this by Archimedian) then: $q > \varepsilon \Rightarrow \frac{1}{q} < \frac{1}{\varepsilon} < \varepsilon \Rightarrow \frac{1}{q} < \varepsilon \Rightarrow \frac{1}{q} - \frac{1}{p} < \varepsilon \Rightarrow -\varepsilon < \frac{1}{p} - \frac{1}{q}$ and $\frac{1}{q} \in$

Hence, $|\frac{1}{q} - \frac{1}{p}| < \varepsilon \Rightarrow d(\frac{1}{p}, \frac{1}{q}) < \varepsilon$. therefore, $\frac{1}{q} \in N_\varepsilon(\frac{1}{p})$; $\frac{1}{q} \neq \frac{1}{p}$; and therefore, $N_\varepsilon(\frac{1}{p}) \setminus \{\frac{1}{p}\} \cap E \neq \emptyset$; so $\frac{1}{p}$ is a limit point of E: for any $p \in$

the argument is symmetric if we reverse the roles of p and q.

What's more, Any other point in E other than $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is not a limit point of E. Let $x \in E$ such that $x \neq \frac{1}{n}$ for $n \in \mathbb{N}$ and $x \neq 0$. Let d be the distance from x to the nearest point $e \in E$. then, let $\varepsilon < \frac{d(x,e)}{2}$ so that $d > 0$ (since $x \neq e$) and $N_\varepsilon(x) \setminus \{x\} \cap E = \emptyset$. Hence x is not a limit point of E.

(b) claim: E' is countable. Pf: this is obvious from the bijection $n \to \frac{1}{n}$; $n \in \mathbb{N}$

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(c) claim: E is compact.

Pf: For this let us show that E is closed and bounded

(i) It is closed since $E' \subset E$, because $\frac{1}{n}$ can be written as:
$\frac{1}{n} = \frac{2}{2n} = \frac{1}{2n} + \frac{1}{2n} \Rightarrow$ choose $p = q = 2n$; for $n \in \mathbb{N}$.

(ii) It is bounded. Pick $R = 10$; $x = 0$; then cleary $N_R(x) = N_{10}(0) = (-5$ and $E \subset N_{10}(x)$. Note that 2 is an upper bound for E and 0 a lower bound

Since E is closed and bounded, $E \subset \mathbb{R} \Rightarrow E$ is compact by theorem 2.41.

7) Let E be the set of all $x \in [0,1]$ whose decimal expansion contains
nly the digits 4 and 7.

(a) Is E countable? Solution: NO. this results follows from the fact
hat we proved in class that two valued sequences are uncountable.
theorem 2.14). Regard the decimal expansion as a two valued sequence, and
ine the bijection $f: E \to \{0\text{-}1 \text{ sequences}\}$; by $f(e) = \{x_i\}$, where $x_i = \begin{cases} 0 & \text{if } e_i = 4 \\ 1 & \text{if } e_i = 7 \end{cases}$
$\Rightarrow E \sim \{\text{binary sequences}\} \Rightarrow E$ is uncountable.

(b) Is E dense in $[0,1]$?
Solution: NO. First note that a number in E must be at least
$\frac{4}{9} = 0.444\cdots$ Any number below $\frac{4}{9}$ will contain a digit less than
in its decimal expansion. therefore, look at $0 \in [0,1]$. Certainly,
$\notin E$ (by the previous argument or just looking at its decimal expansion
$0000\cdots \notin E$). Take a neighborhood of 0 with a radius $r < \frac{4}{9}$ (or to
sure take $r < \frac{4/9}{1000}$, → just an arbitrary big number) so that
$N_r(0) \setminus \{0\} \cap E = \emptyset$, in pictures

. therefore, 0 is not
limit point of E. So, E is not dense in $[0,1]$.

Is E compact?
Solution: YES. Clearly E is bounded. take $R = 2$, $x = 0$, $N_2(0) = (-2,2)$
$\subset [0,1] \subset [-2,2] \Rightarrow E \subset [-2,2]$.
Moreover, E is closed. To prove this, let us show that E^c is open.
$x \in E^c \Rightarrow x \notin E$; so x has at least one digit in its decimal expansion
being not 4 or 7. $x = x_1x_2x_3\cdots x_m x_{m+1}\cdots$; Let x_m be the first such place.
$r = \frac{1}{10^{m+1}}$. then $N_r(x) \subset E^c \Rightarrow x$ is interior to $E^c \Rightarrow E^c$ is open
E is closed. Since E is closed, bounded and $E \subset \mathbb{R} \Rightarrow E$ is compact.

Is E perfect?
Solution: YES. We already proved that E is closed. It remains to prove
that all points in E are limit points of E, in other words E has no
isolated points. let $r > 0$, and $x \in E$. then $N_r(x) \setminus \{x\} \cap E \neq \emptyset$, because
choosing m so that $r > \frac{1}{10^m}$; then numbers in $N_r(x)$ will agree with x
at least on the first m decimal places; so $N_r(x) \setminus \{x\}$ contains points in E
any $r > 0$, any $x \in E$ is a limit point of E. thus, E is perfect.

10