M403 Homework 9

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October 31, 2012

(2.1) (i) **True.** By hypothesis if $x \in S$ then $x \in T$. Also, if $x \in T$ then $x \in X$. Hence, if $x \in S$ then $x \in X \iff S \subseteq X$

(ii) **False.** Since $f \circ g$ is not a well-defined function.

(iii) **True.** Since $g \circ f : X \to Z$ is a well-defined function.

(iv) **True.** Because by definition $X \times \emptyset \{(x, y) : x \in X \text{ and } y \in \emptyset\}$, but nothing belong to the empty set. Hence, $X \times \emptyset = \emptyset$

(v) **True.** Let $h : imgf \to imgf$ be defined as $h(y) = y$ for all $y \in imgf$. Then $g : X \to imgf$ defined as $g = h \circ f$ is a surjection since for every $y \in imgf$ there is an $x \in X$ such that $y = f(x)$ (by definition of $imgf$). Moreover, since $j \circ g : X \to Y$ and $f : X \to Y$ and $f(x) = (j \circ g)(x) = j(g(x)) = j(y) = y$, then $j \circ g = f$.

(vi) **False.** let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(n) = |n|$ and $g : \mathbb{N} \to \mathbb{Z}$ be defined as $g(n) = n$. Then, $f \circ g : \mathbb{N} \to \mathbb{N}$ is a well-defined function such that $f \circ g = 1_{\mathbb{N}}$, since: $(f \circ g)(n) = f(g(n)) = f(n) = |n| = n$. However, f is not a bijection since it is not injective. Indeed, $f(2) = f(-2) = 2$ but $2 \neq -2$.

(vii) **False.** Since $\frac{1}{2}, \frac{2}{4} \in \mathbb{Q}$ are such that $\frac{1}{2} = \frac{2}{4}$, but $f(\frac{1}{2}) = 1^2 - 2^2 = 1 - 4 = -3 \neq f(\frac{2}{4}) = 2^2 - 4^2 = 4 - 16 = -12$.

(viii) **False.** since $(g \circ f)(n) = g(f(n)) = g(n+1) = (n+1)^2 = n^2 + 2n + 1 \neq n(n+1)$

(ix) **True.** The map $conj : \mathbb{C} \to \mathbb{C}$ defined as $conj(a + ib) = a - ib$ is a bijection. It is <u>injective</u> since given $a_1 + ib_1, a_2 + ib_2 \in \mathbb{C}$, if $conj(a_1 + ib_1) = conj(a_2 + ib_2)$ then $a_1 - ib_1 = a_2 - ib_2$ which by equality of complex numbers means that $a_1 = a_2$ and $b_1 = b_2$ and hence, $a_1 + ib_1 = a_2 + ib_2$. It is <u>surjective</u> since for any $a + ib \in \mathbb{C}$ we can always take $a - ib \in \mathbb{C}$ such that $f(a - ib) = ai + b$. Since $conj$ is both injective and surjective it is also bijective.

(2.3) (i) <u>Proof of:</u> $(A \cup B)' = A' \cap B'$

Let $x \in (A \cup B)'$	$\iff$	$x \in X - (A \cup B)$	By definition of set complement
	$\iff$	$x \in X$ and $x \notin A \cup B$	By definition of set difference
	$\iff$	$x \in X$ and $x \notin A$ and $x \notin B$	By definition of membership in the union
	$\iff$	$(x \in X$ and $x \notin A)$ and $(x \in X$ and $x \notin B)$	Grouping statements
	$\iff$	$x \in X - A$ and $x \in X - B$	By definition of set difference
	$\iff$	$x \in A'$ and $x \in B'$	By definition of set complement
	$\iff$	$x \in A' \cap B'$	By definition of intersection

(ii) <u>Proof of:</u> $(A \cap B)' = A' \cup B'$

Let $x \in (A \cap B)'$	$\iff$	$x \in X - (A \cap B)$	By definition of set complement
	$\iff$	$x \in X$ and $x \notin A \cap B$	By definition of set difference
	$\iff$	$x \in X$ and $(x \notin A$ or $x \notin B)$	By definition of membership in the inter.
	$\iff$	$(x \in X$ and $x \notin A)$ or $(x \in X$ and $x \notin B)$	Grouping statements
	$\iff$	$x \in X - A$ or $x \in X - B$	By definition of set difference
	$\iff$	$x \in A'$ or $x \in B'$	By definition of set complement
	$\iff$	$x \in A' \cup B'$	By definition of union

(2.7) (i) Let $S \subseteq X$ and $f : X \to Y$. First note that by definition $f|S : S \to Y$ and $f \circ i : S \to Y$. Given $s \in S$:

$$\begin{aligned} (f|S)(s) &= f(s) && \text{by definition of } f|S \\ &= f(i(s)) && \text{by definition } i(s) = s \text{ for all } s \in S \\ &= (f \circ i)(s) && \text{function composition} \end{aligned}$$

Hence, for any $s \in S$, $f|S(s) = f \circ i(s)$. Therefore, $f|S = f \circ i$

(ii) Consider the function $f' : X \to A$ defined as $f' = j' \circ f$, where $j' : Y \to A$ defined as $j'(y) = y$. The claim is that f' is a surjection since by hypothesis $im(f) = A \subseteq Y$ and so every element of the codomain of f' has a preimage. Moreover, if we consider the function $j \circ f' : X \to Y$, where $j : A \to Y$ is the inclusion, i.e., $j(a) = a$ for all $a \in A$, then we can conclude that $j \circ f' = j \circ (j' \circ f) = (j \circ j') \circ f = 1_Y \circ f = f$

(2.9) Suppose that $f : X \to Y$ is a bijection with two inverses, i.e., there exists two functions $f_1 : Y \to X$ and $f_2 : Y \to X$ such that:

$$f \circ f_1 = 1_Y \text{ and } f_1 \circ f = 1_X$$
$$f \circ f_2 = 1_Y \text{ and } f_2 \circ f = 1_X$$

But then:

$$\begin{aligned} f_1 &= f_1 \circ 1_Y && \text{since } 1_Y \text{ is the identity function} \\ &= f_1 \circ (f \circ f_2) && \text{by definition of } 1_Y \\ &= (f_1 \circ f) \circ f_2 && \text{since function composition is associative} \\ &= 1_X \circ f_2 && \text{by definition of } 1_X \\ &= f_2 && \text{since } 1_X \text{ is the identity function} \end{aligned}$$

Hence, $f_1 = f_2$, which means that the inverse of f is unique.

(2.13) (i) We need only to show that f is injective if and only if f is surjective. The other implications follow from this, i.e., f bijective $\iff$ f injective and f surjective.

$(i) \Rightarrow (iii)$ Suppose f is injective. Suppose by way of contradiction that f is not surjective. Then $imgf \subset Y$ which means that $|imgf| < |Y| = n$. Let $p = |imgf|$, then $p < n$. Take p different elements of X, say $x_1, x_2, ..., x_p$. Apply f to these elements to obtain $f(x_1), f(x_2), ..., f(x_p) \in imgf$ by definition. Since f is injective, $f(x_i) \neq f(x_j)$ for all $1 \leq i, j \leq p$ where $i \neq j$ and hence, $imgf = \{f(x_1), f(x_2), ..., f(x_p)\}$. But since we only considered p elements in X and $|X| = n > p$, we know that there exists an element $x \in X$ such that $x \notin \{x_1, x_2, ..., x_p\}$. Apply f to this element to obtain $f(x) \in imgf$ by definition of image. So $f(x) = f(x_i)$ for $1 \leq i \leq p$, which contradicts the fact that f is injective. Therefore, f must be surjective.

$(iii) \Rightarrow (i)$ Suppose f is surjective. Suppose by way of contradiction that f is not injective. Then, there exists $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. Construct the set $S = \{x_1, x_2\} \subset X$. By definition of set complementation we have that $X = S \cup S'$, where S' is the complement of S. Now, apply f to both sides of the equation: $f(X) = f(S \cup S')$. By definition of image we have that $imgf = f(S \cup S')$. Since f is surjective, $imgf = Y$ and thus, $Y = f(S \cup S')$. By exercise 2.16 (i), $f(S \cup S') = f(S) \cup f(S')$, from which we conclude that $Y = f(S) \cup f(S') \Rightarrow$ since S and S' are disjoint sets: $|Y| = |f(S)| + |f(S')|$. But by constructing $|f(S)| = 1$ and $|f(S)| \leq n - 2$. Hence, $n = |Y| \leq 1 + n - 2 = n - 1$, a contradiction. Therefore, f is injective.

(ii) Let P be the set of pigeons. $|P| = 11$. Let H be the set of holes. $|H| = 10$. Define the map $sits : P \to H$, that assigns to each pigeon in the set P a hole in the set H. Since both $|P|$ and $|H|$ are finite sets such that $|P| \neq |H|$, there exists no possible bijection between them. Since $|P| > |H|$ and by (i), this means that the map $sits$ is not injective, i.e., there exists $p_1, p_2 \in P$ such that $sits(p_1) = sits(p_2)$ and $p_1 \neq p_2$. This is the same as stating that there is one hole containing more than one pigeon (two different pigeons sitting on the same hole).

(2.14) (i) Suppose both f and g are injective. Suppose also that, given $x_1, x_2 \in X$, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then:

$$\begin{aligned} (g \circ f)(x_1) = (g \circ f)(x_2) &\iff g(f(x_1)) = g(f(x_2)) && \text{By definition of function composition} \\ &\implies f(x_1) = f(x_2) && \text{Since } g \text{ is injective and } f(x_1), f(x_2) \in Y \\ &\implies x_1 = x_2 && \text{Since } f \text{ is injective} \end{aligned}$$

Hence, $g \circ f$ is injective.

(ii) Suppose both f and g are surjective. Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Moreover, since f is surjective, there exists $x \in X$ such that $y = f(x)$. Replacing this equation into our previous equation we obtain that $g(f(x)) = z \iff (g \circ f)(x) = z$. Therefore, given any $z \in Z$, we can always find a $x \in X$ such that $(g \circ f)(x) = z$ which means that $g \circ f$ is surjective.

(iii) If both f, g are bijective then by definition both f, g are injective and surjective. In (i) we proved that if both f, g are injective then $g \circ f$ is also injective. In (ii) we proved that if f, g are surjective then $g \circ f$ is also surjective. Therefore, if we assume that both f, g are bijective, then we can conclude that f, g are both injective and surjective which by the aforementioned reasons means that $g \circ f$ is bijective as well.

(iv) Suppose $g \circ f$ is a bijection. This means that $g \circ f$ is injective and surjective.

Proof f is injective: Let $x_1, x_2 \in X$. Suppose that $f(x_1) = f(x_2)$. We can apply g to both sides of this equation to obtain $g(f(x_1)) = g(f(x_2))$, which by definition is the same as $(g \circ f)(x_1) = (g \circ f)(x_2)$. But, since $g \circ f$ is injective, we conclude that $x_1 = x_2$, which means that f is injective.

Proof g is surjective: Let $z \in Z$. Since $g \circ f$ is a surjection, given $z \in Z$, there exists $x \in X$ such that $(g \circ f)(x) = z \iff g(f(x)) = z$. Since $f(x) \in Y$, call it $y = f(x)$. Then, for any element $z \in Z$ it is true that there exists y such that $g(y) = z$, just take $y = f(x)$. Hence, g is a surjection.