# M403 Homework 11

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(2.21) (i) **False.** Let $n = 3$. Then $|S_3| = 3! = 6 > 3 = n$.

(ii) **True.** We can write $\sigma$ as a product of cycles. Then $n = lcm$ of the lengths of all cycles.

(iii) **True.** This is the standard notation of composition of permutations as product.

(iv) **False.** Let $\alpha = \begin{pmatrix} 3 & 4 \end{pmatrix} \in S_4$ and $\beta = \begin{pmatrix} 4 & 2 \end{pmatrix} \in S_4$. Then

$$\alpha\beta = \begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

Which is not the same as:

$$\beta\alpha = \begin{pmatrix} 4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

(v) **False.** Let $\alpha$ and $\beta$ be as before. Both $\alpha$ and $\beta$ are 2-cycles. But:

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$$

Which is a 3-cycle.

(vi) **True.** Consequence of proposition 2.33.

(x) **False.** Let $\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \in S_4$. Then, $\sigma^{-1} = \begin{pmatrix} 2 & 1 \end{pmatrix} \in S_4$.
Let $\omega = \begin{pmatrix} 3 & 4 \end{pmatrix} \in S_4$. Then, $\omega^{-1} = \begin{pmatrix} 4 & 3 \end{pmatrix} \in S_4$. Let $\alpha = \begin{pmatrix} 3 & 4 \end{pmatrix} \in S_4$. Then $\sigma \neq \omega$, but

$$\sigma\alpha\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} =$$

$$\omega\alpha\omega^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

(2.22) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \end{pmatrix}\begin{pmatrix} 8 & 2 \end{pmatrix}\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 4 & 6 \end{pmatrix}\begin{pmatrix} 5 \end{pmatrix}$. The inverse is:
$\alpha^{-1} = \begin{pmatrix} 5 \end{pmatrix}\begin{pmatrix} 6 & 4 \end{pmatrix}\begin{pmatrix} 7 & 3 \end{pmatrix}\begin{pmatrix} 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 9 \end{pmatrix}$. We can verify this:

$$\alpha\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

(2.24) (i) Let $1 < r \leq n$. An $r$-cycle is of the form $(\sigma(1)\ \sigma(2)\ \cdots\ \sigma(r))$. There are $n$ choices for $\sigma(1)$, n-1 choices for $\sigma(2)$,..., and finally $n - r + 1$ choices for $\sigma(r)$. By the rule of product, there are $n(n-1)\cdots(n-r+1)$ total choices. However, we regard circular orders as being the same so we must divide this expression by $r$, i.e., $[n(n-1)\cdots(n-r+1)]\frac{1}{r}$ is the total number of $r$-cycles in $S_n$.

(ii) The proof is by <u>induction</u>. Consider the following statement $S(k)$ : the number of permutations $\alpha \in S_n$, where $\alpha$ is a product of $k$ disjoint $r$-cycles is $\frac{1}{k!}\frac{1}{r^k}[n(n-1)\cdots(n-r+1)]$.

<u>Base Case:</u> $S(1)$ is true since we proved it in (i).

<u>Inductive Step:</u> Assume $S(k-1)$ is true. We want to show that $S(k)$ is true. Let $\alpha \in S_n$ be a product of $k$ disjoint $r-$cycles. Then, we can write alpha as $\alpha = \sigma\beta$, where $\sigma$ is a product of $k-1$ disjoint cycles and $\beta$ is an $r-$cycle. Then, we have $n - r(k-1)$ choices for $\beta$, but we need to divide by $k$ to account for linear combinations. Hence, the total number of permutations of products of $k$ disjoint $r$-cycles is:

$$\frac{1}{k}\frac{1}{r}\left(\frac{(n-r(k-1)!}{(n-kr)!}\right)\frac{1}{(k-1)!}\frac{1}{r^{k-1}}\frac{n!}{(n-(k-1)r)!} = S(k)$$

(2.25) (i) Let $\alpha$ be an r-cycle. Then:

$$\begin{array}{rcll} \alpha & = & ( i_1 \ i_2 \ \cdots \ i_r ) & \text{By definition} \\ \alpha^r & = & [( i_1 \ i_2 \ \cdots \ i_r )]^r & \text{Raising } \alpha \text{ to the r power.} \\ & = & ( i_1 \ i_2 \ \cdots \ i_r ) \cdots ( i_1 \ i_2 \ \cdots \ i_r ) ( i_1 \ i_2 \ \cdots \ i_r ) & \text{By definition of exponentiation.} \\ & = & ( i_1 \ i_2 \ \cdots \ i_r ) \cdots ( i_1 \ i_2 \ \cdots \ i_r ) [( i_r \ i_1 \ \cdots \ i_{r-1} )] & \text{Operating the last two terms} \\ & \vdots & & \\ & = & ( i_1 \ i_2 \ \cdots \ i_r ) ( i_r \ i_{r-1} \ \cdots \ i_1 ) & \text{Operating } r-1 \text{ terms} \\ & = & (1) & \text{By definition of inverse} \end{array}$$

(ii) It follows from the previous proof that if $\alpha$ is an r-cycle, any positive integer $k < r$ is such that $\alpha^k \neq (1)$ and $\alpha^r = (1)$. Hence, $r$ is the least positive integer such that $\alpha^r = (1)$

(2.33) Let $\alpha = ( 1 \ 2 ), \beta = ( 3 \ 4 ), \gamma = ( 3 \ 5 ) \in S_5$ none of which is the identity and, since $\alpha$ and $\beta$, and $\alpha$ and $\gamma$ are disjoints transpositions, we have that:

$$\alpha\beta = ( 1 \ 2 ) ( 3 \ 4 ) = ( 3 \ 4 ) ( 1 \ 2 ) = \beta\alpha$$

$$\alpha\gamma = ( 1 \ 2 ) ( 3 \ 4 ) = ( 3 \ 4 ) ( 1 \ 2 ) = \beta\alpha$$

But,

$$\beta\gamma = ( 3 \ 4 ) ( 3 \ 5 ) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{pmatrix} \neq$$

$$\gamma\beta = ( 3 \ 5 ) ( 3 \ 4 ) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

I) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 7 & 2 & 1 & 4 & 3 \end{pmatrix} = ( 1 \ 5 \ 2 \ 6 ) ( 3 \ 8 ) ( 4 \ 7 ) \Rightarrow \alpha^{-1} = ( 7 \ 4 ) ( 8 \ 3 ) ( 6 \ 2 \ 5 \ 1 ).$

In double-row notation: $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 1 & 2 & 4 & 3 \end{pmatrix}$

II) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 8 & 7 & 5 & 4 & 6 \end{pmatrix} = ( 1 \ 3 \ 2 ) ( 4 \ 8 \ 6 \ 5 \ 7 ) \Rightarrow \alpha^{-1} = ( 7 \ 5 \ 6 \ 8 \ 4 ) ( 2 \ 3 \ 1 ).$

In double-row notation: $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 7 & 6 & 8 & 5 & 4 \end{pmatrix}$

III) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 5 & 6 & 7 & 8 & 1 \end{pmatrix} = ( 1 \ 3 \ 2 \ 4 \ 5 \ 6 \ 7 \ 8 ) \Rightarrow \alpha^{-1} = ( 8 \ 7 \ 6 \ 5 \ 4 \ 2 \ 3 \ 1 ).$

In double-row notation: $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 1 & 2 & 4 & 5 & 6 & 7 \end{pmatrix}$

IV) $\alpha = ( 1 \ 2 ) ( 2 \ 3 ) ( 3 \ 4 ) ( 4 \ 5 ) ( 5 \ 6 ) \Rightarrow \alpha^{-1} = ( 6 \ 5 ) ( 5 \ 4 ) ( 4 \ 3 ) ( 3 \ 2 ) ( 2 \ 1 )$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 6 & 1 & 7 & 8 & 9 & 10 \end{pmatrix} \Rightarrow \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 \end{pmatrix}$$

V) $\alpha = ( 1 \ 2 \ 3 \ 4 \ 5 ) ( 5 \ 6 \ 7 \ 8 ) \Rightarrow \alpha^{-1} = ( 8 \ 7 \ 6 \ 5 ) ( 5 \ 4 \ 3 \ 2 \ 1 )$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 9 & 10 \end{pmatrix} \Rightarrow \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 \end{pmatrix}$$

VI) $\alpha = ( 1 \ 5 \ 9 ) ( 2 \ 6 \ 10 ) ( 4 ) \Rightarrow \alpha^{-1} = ( 4 ) ( 10 \ 6 \ 2 ) ( 9 \ 5 \ 1 )$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 6 & 3 & 4 & 9 & 10 & 7 & 8 & 1 & 2 \end{pmatrix} \Rightarrow \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 10 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix}$$