Math 396. Quotient spaces

1. Definition

Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$ and $W \subseteq V$ a subspace of V. For $v_1, v_2 \in V$, we say that $v_1 \equiv v_2 \bmod W$ if and only if $v_1 - v_2 \in W$. One can readily verify that with this definition congruence modulo W is an equivalence relation on V. If $v \in V$, then we denote by $\overline{v} = v + W = \{v + w : w \in W\}$ the equivalence class of v. We define the *quotient space* of V and W as $V/W = \{\overline{v} : v \in V\}$, and we make it into a vector space over $\mathbb{F}$ with the operations $\overline{v_1} + \overline{v_2} = \overline{v_1 + v_2}$ and $\lambda\overline{v} = \overline{\lambda v}$, $\lambda \in \mathbb{F}$. The proof that these operations are well-defined is similar to what is needed in order to verify that addition and multiplication are well-posed operations on $\mathbb{Z}_n$, the integers mod n, $n \in \mathbb{N}$. Lastly, define the *projection* $\pi\colon V \to V/W$ as $v \mapsto \overline{v}$. Note that π is both linear and surjective.

One can, but in general should not, try to visualize the quotient space V/W as a subspace of the space V. With this in mind, in Figure 1 we have a diagram of how one might do this with $V = \mathbb{R}^2$ and $W = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ (the x-axis). Note that $(x, y) \equiv (\tilde{x}, \tilde{y}) \mod W$ if and only if $y = \tilde{y}$. Equivalence classes of vectors (x, y) are of the form $\overline{(x, y)} = (0, y) + W$. A few of these are indicated in Figure 1 by dotted lines. Hence one could linearly identify $\mathbb{R}^2/W$ with the y-axis. Similarly, if L is any line in $\mathbb{R}^2$ distinct from W then $L \oplus W = \mathbb{R}^2$ and so consequently every element in $\mathbb{R}^2$ has a unique representative from L modulo W, so the composite map $L \to \mathbb{R}^2/W$ is an isomorphism. Of course, there are many choices of L and none is better than any other from an algebraic point of view (though the y-axis may look like the nicest since it is the orthogonal complement to the x-axis W, a viewpoint we will return to later).

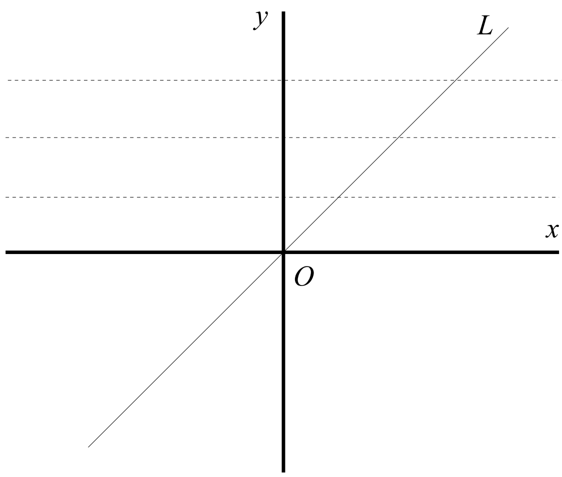


Figure 1. The quotient $\mathbb{R}^2/W$

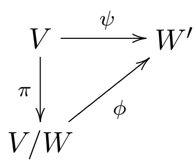
After looking at Figure 1, one should forget it as quickly as possible. The problems with this sort of identification are discussed later on. Roughly speaking, there are many natural operations associated with the quotient space that are not compatible with the above non-canonical view of the quotient space as a subspace of the original vector space. The issues are similar to looking at $\{0, \ldots, 9\} \subseteq \mathbb{Z}$ as representative of $\mathbb{Z}_{10}$. For example, a map $f\colon \mathbb{Z} \to \mathbb{Z}$, $z \mapsto 2z$ cannot be thought of in the same way as the map $\overline{z} \mapsto \overline{2z}$ because $f(\{0, \ldots, 9\}) \not\subseteq \{0, \ldots, 9\}$. Although one should not view $\{0, \ldots, 9\}$ as a fixed set of elements lying in $\mathbb{Z}$, just as one should not in general identify

V/W with a fixed set of representatives in V, this view can be useful for various computations. We will see this is akin to the course of action that we will take in the context of quotient spaces in a later section.

2. Universal Property of the Quotient

Let $\mathbb{F}, V, W$ and π be as above. Then the quotient V/W has the following universal property: *Whenever W' is a vector space over $\mathbb{F}$ and $\psi\colon V \to W'$ is a linear map whose kernel contains W, then there exists a unique linear map $\phi\colon V/W \to W'$ such that $\psi = \phi \circ \pi$.* The universal property can be summarized by the following commutative diagram:

(1)



Proof. The proof of this fact is rather elementary, but is a useful exercise in developing a better understanding of the quotient space. Let W' be a vector space over $\mathbb{F}$ and $\psi\colon V \to W'$ be a linear map with $W \subseteq \ker(\psi)$. Then define $\phi\colon V/W \to W'$ to be the map $\overline{v} \mapsto \psi(v)$. Note that ϕ is well defined because if $\overline{v} \in V/W$ and $v_1, v_2 \in V$ are both representatives of $\overline{v}$, then there exists $w \in W$ such that $v_1 = v_2 + w$. Hence,

$$\psi(v_1) = \psi(v_2 + u) = \psi(v_2) + \psi(w) = \psi(v_2).$$

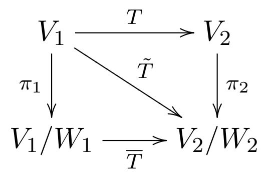
The linearity of ϕ is an immediate consequence of the linearity of ψ. Furthermore, it is clear from the definition that $\psi = \phi \circ \pi$. It remains to show that ϕ is unique. Suppose that $\sigma\colon V/W \to W'$ is such that $\sigma \circ \pi = \psi = \phi \circ \pi$. Then $\sigma(\overline{v}) = \phi(\overline{v})$ for all $v \in V$. Since π is surjective, $\sigma = \phi$. □

As an example, let $V = C^\infty(\mathbb{R})$ be the vector space of infinitely differentiable functions over the field $\mathbb{R}$, $W \subseteq V$ the space of constant functions and $\varphi\colon V \to V$ the map $f \mapsto f'$, the differentiation operator. Then clearly $W \subseteq \ker(\varphi)$ and so there exists a unique map $\phi\colon V/W \to V$ such that $\varphi = \phi \circ \pi$, $\pi\colon V \to V/W$ the projection. Two functions $f, g \in V$ satisfy $\overline{f} = \overline{g}$ if and only if $f' = g'$. Note that the set of representatives for each $\overline{f} \in V/W$ can be thought of as $f + \mathbb{R}$. Concretely, the content of this discussion is that the computation of the derivative of f depends only on f up to a constant factor.

In a sense, all surjections “are” quotients. Let $T\colon V \to V'$ be a surjective map with kernel $W \subseteq V$. Then there is an induced linear map $\overline{T}\colon V/W \to V'$ that is surjective (because T is) and injective (follows from def of W). Thus to factor a linear map $\psi\colon V \to W'$ through a surjective map T is the “same” as factoring ψ through the quotient V/W.

One can use the univeral property of the quotient to prove another useful factorization. Let V_1 and V_2 be vector spaces over $\mathbb{F}$ and $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ be subspaces. Denote by $\pi_1\colon V_1 \to V_1/W_1$ and $\pi_2\colon V_2 \to V_2/W_2$ the projection maps. Suppose that $T\colon V_1 \to V_2$ is a linear map with $T(W_1) \subseteq W_2$ and let $\tilde{T}\colon V_1 \to V_2/W_2$ be the map $x \mapsto \pi_2(T(x))$. Note that $W_1 \subseteq \ker(\tilde{T})$. Hence by the universal property of the quotient, there exists a unique linear map $\overline{T}\colon V_1/W_1 \to V_2/W_2$ such that $\overline{T} \circ \pi_1 = \tilde{T}$. The contents of this statement can be summarized by saying that if everything is as above, then the following diagram commutes:

(2)



In such a situation, we will refer to $\overline{T}$ as the *induced map.* The purpose of this is that for $v_1 \in V_1$, the class of $T(v_1)$ modulo W_2 depends only on the class of v_1 modulo W_1, and that the resulting association of congruence classes given by $\overline{T}$ is linear with respect to the linear structure on these quotient spaces. The induced map comes to be rather useful as a tool for inductive proofs on the dimension. We will see an example of this at the end of this handout.

3. A Basis of the Quotient

Let V be a finite-dimensional $\mathbb{F}$-vector space (for a field $\mathbb{F}$), W a subspace, and

$$\{w_1, \ldots, w_m\}, \{w_1, \ldots, w_m, v_1, \ldots, v_n\}$$

ordered bases of W and V respectively. We claim that the image vectors $\{\overline{v}_1, \ldots, \overline{v}_n\}$ in V/W are a basis of this quotient space. In particular, this shows $\dim(V/W) = \dim V - \dim W$.

The first step is to check that all $\overline{v}_j$'s span V/W, and the second step is to verify their linear independence. Choose an element $x \in V/W$, and pick a representative $v \in V$ of x (i.e., $V \twoheadrightarrow V/W$ sends $v \in V$ onto our chosen element, or in other words $x = \overline{v}$). Since $\{w_i, v_j\}$ spans V, we can write

$$v = \sum a_i w_i + \sum b_j v_j.$$

Applying the *linear* projection $V \twoheadrightarrow V/W$ to this, we get

$$x = \overline{v} = \sum a_i \overline{w}_i + \sum b_j \overline{v}_j = \sum b_j \overline{v}_j$$

since $\overline{w}_i \in V/W$ vanishes for all i. Thus, the $\overline{v}_j$'s span V/W.

Meanwhile, for linear independence, if $\sum b_j \overline{v}_j \in V/W$ vanishes for some $b_j \in \mathbb{F}$ then we want to prove that all b_j vanish. Well, consider the element $\sum b_j v_j \in V$. This projects to 0 in V/W, so we conclude $\sum b_j v_j \in W$. But W is spanned by the w_i's, so we get

$$\sum b_j v_j = \sum a_i w_i$$

for some $a_i \in \mathbb{F}$. This can be rewritten as

$$\sum b_j v_j + \sum (-a_i) w_i = 0$$

in V. By the assumed linear independence of $\{w_1, \ldots, w_m, v_1, \ldots, v_n\}$ we conclude that all coefficients on the left side vanish in $\mathbb{F}$. In particular, all b_j vanish.

Note that there is a converse result as well: if $v_1, \ldots, v_n \in V$ have the property that their images $\overline{v}_i \in V/W$ form a basis (so $n = \dim V - \dim W$), then for any basis $\{w_j\}$ of W the collection of v_i's and w_j's is a basis of V. Indeed, since $n = \dim V - \dim W$ then the size of the collection of v_i's and w_j's is $\dim V$, and hence it suffices to check that they span. For any $v \in V$, its image $\overline{v} \in V/W$ is a linear combination $\sum a_i \overline{v}_i$, so $v - \sum a_i v_i \in V$ has image 0 in V/W. That is, $v - \sum a_i v_i \in W$, and so $v - \sum a_i v_i = \sum b_j w_j$ for some b_j's. This shows that v is in the span of the v_i's and w_j's, as desired.

4. Some calculations

We now apply the above principles to compute matrices for an induced map on quotients using several different choice of bases. Let $A : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map described by the matrix

$$\begin{pmatrix} 2 & 0 & -3 & 4 \\ 1 & 1 & 0 & -2 \\ 3 & -5 & 7 & 6 \end{pmatrix}$$

Let $L \subseteq \mathbb{R}^4$ and $L' \subseteq \mathbb{R}^3$ be the lines:

$$L = \text{span}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad L' = \text{span}\begin{pmatrix} -4 \\ 0 \\ 22 \end{pmatrix}$$

We wish to do two things:

(i) In the specific setup at the start, check that A sends L into L'. Letting $\overline{A} : \mathbb{R}^4/L \to \mathbb{R}^3/L'$ denote the induced linear map, if $\{e_i\}$ and $\{e'_j\}$ denote the respective standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$ then we wish to compute the matrix of $\overline{A}$ with respect to the induced bases $\{\overline{e}_2, \overline{e}_3, \overline{e}_4\}$ of $\mathbb{R}^4/L$ and $\{\overline{e}'_1, \overline{e}'_2\}$ of $\mathbb{R}^3/L'$. The consideration given in the previous section ensures that the $\overline{e}$'s form bases of their respective quotient spaces. Indeed, the e's along with the vector spanning their corresponding lines do in fact form a bases of their respective ambient vector spaces.

(ii) We want to compute the matrix for $\overline{A}$ if we switch to the basis $\{\overline{e}_1, \overline{e}_2, \overline{e}_4\}$ of $\mathbb{R}^4/L$, and check that this agrees with the result obtained from using our answer in (i) and a 3 by 3 change-of-basis matrix to relate the ordered bases $\{\overline{e}_2, \overline{e}_3, \overline{e}_4\}$ and $\{\overline{e}_1, \overline{e}_2, \overline{e}_4\}$ of $\mathbb{R}^4/L$. We also want to show that $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$ is *not* a basis of the 3-dimensional $\mathbb{R}^4/L$ by explicitly exhibiting a vector $x \in \mathbb{R}^4/L$ which is not in the span of this triple (and *prove* that this x really is not in the span).

Solution

(i) It is trivial to check that A sends the indicated spanning vector of L to exactly the indicated spanning vector of L', so by linearity it sends the line L over into the line L'. Thus, we get a well-defined linear map $\overline{A} : \mathbb{R}^4/L \to \mathbb{R}^3/L'$ characterized by

$$\overline{A}(v \bmod L) = A(v) \bmod L'.$$

Letting w denote the indicated spanning vector of L, the 4-tuple $\{e_2, e_3, e_4, w\}$ is a basis of $\mathbb{R}^4$ is a linearly independent 4-tuple (since w has non-zero first coordinate!) in a 4-dimensional space, and hence it is a basis. Thus, by the basis criterion for quotients, $\{\overline{e}_2, \overline{e}_3, \overline{e}_4\}$ is a basis of $\mathbb{R}^4/L$. By a similar argument, since the indicated spanning vector of L' has non-zero third coordinate, it follows that $\{\overline{e}'_1, \overline{e}'_2\}$ is a basis of $\mathbb{R}^3/L'$.

To compute the matrix ${}_{\{\overline{e}'_1, \overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_2, \overline{e}_3, \overline{e}_4\}}$, we have to find the (necessarily unique) elements $a_{ij} \in \mathbb{R}$ such that

$$\overline{A}(\overline{e}_j) = a_{1j}\overline{e}'_1 + a_{2j}\overline{e}'_2$$

in $\mathbb{R}^3/L'$ (or equivalently, $A(e_j) \equiv a_{1j}e'_1 + a_{2j}e'_2 \bmod L'$) and then the sought-after matrix will be

$$\begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \end{pmatrix}$$

Note that the labelling of the a_{ij}'s here is chosen to match that of the $\overline{e}_j$'s and the $\overline{e}'_i$'s, and of course has nothing to do with labelling "positions" within a matrix. We won't use such notation when actually doing calculations with this matrix, for at such time we will have actual numbers in the various slots and hence won't have any need for generic matrix notation.

Now for the calculation. We must express $\overline{A}(\overline{e}_j)$ as a linear combination of $\overline{e}'_1$ and $\overline{e}'_2$. Well, by definition we are given $A(e_j)$ as a linear combination of e'_1, e'_2, e'_3. Working *modulo* L' we want to express such things in terms of e'_1 and e'_2, so the issue is to express $e'_3 \bmod L'$ as a linear combination of $e'_1 \bmod L'$ and $e'_2 \bmod L'$. That is, we seek $a, b \in \mathbb{R}$ such that

$$e'_3 \equiv ae'_1 + be'_2 \bmod L',$$

and then we'll substitute this into our formulas for each $A(e_j)$.

To find a and b, we let

$$w' = \begin{pmatrix} -4 \\ 0 \\ 22 \end{pmatrix}$$

be the indicated basis vector of L', and since $\{e'_1, e'_2, w'\}$ is a basis of $\mathbb{R}^3$ we know that there is a unique expression

$$e'_3 = ae'_1 + be'_2 + cw'$$

and we merely have to make this explicit (for then $e'_3 \equiv ae'_1 + be'_2 \bmod L'$). Writing out in terms of elements of $\mathbb{R}^3$, this says

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a - 4c \\ b \\ 22c \end{pmatrix},$$

so we easily solve $c = 1/22$, $b = 0$, $a = 4c = 4/22 = 2/11$.

That is, $e'_3 \equiv (2/11)e'_1 \bmod L'$, or in other words

$$\overline{e}'_3 = \frac{2}{11}\overline{e}'_1$$

in $\mathbb{R}^3/L'$. Thus, going back to the definition of A, we compute

$$\overline{A}(\overline{e}_2) = \overline{e}'_2 - 5\overline{e}'_3 = -\frac{10}{11}\overline{e}'_1 + \overline{e}'_2,$$

$$\overline{A}(\overline{e}_3) = -3\overline{e}'_1 + 7\overline{e}'_3 = -\frac{19}{11}\overline{e}'_1,$$

$$\overline{A}(\overline{e}_4) = 4\overline{e}'_1 - 2\overline{e}'_2 + 6\overline{e}'_3 = \frac{56}{11}\overline{e}'_1 - 2\overline{e}'_2,$$

so

$$_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_2,\overline{e}_3,\overline{e}_4\}} = \begin{pmatrix} -\frac{10}{11} & -\frac{19}{11} & \frac{56}{11} \\ 1 & 0 & -2 \end{pmatrix}$$

(*ii*) Since the third coordinate of the indicated spanning vector w of L is non-zero, we see that $\{e_1, e_2, e_4, w\}$ is a basis of $\mathbb{R}^4$, so $\{\overline{e}_1, \overline{e}_2, \overline{e}_4\}$ is a basis of $\mathbb{R}^4/L$. The "hard part" of the calculation was already done in (*i*) where we computed $\overline{e}'_3$ as a linear combination of $\overline{e}'_1$ and $\overline{e}'_2$:

$$e'_3 \equiv \frac{2}{11}e'_1 \bmod L'$$

Using this, we compute

$$A(e_1) = 2e'_1 + e'_2 + 3e'_3 \equiv \frac{28}{11}e'_1 + e'_2 \bmod L',$$

so in conjunction with the computations of $A(e_2)$ and $A(e_4)$ modulo L' which we did in (*ii*), we get

$$_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_1,\overline{e}_2,\overline{e}_4\}} = \begin{pmatrix} \frac{28}{11} & -\frac{10}{11} & \frac{56}{11} \\ 1 & 1 & -2 \end{pmatrix}$$

The relevant change of basis matrices for passing between our two coordinate systems are

$$M\colon={}_{\{\overline{e}_2,\overline{e}_3,\overline{e}_4\}}[\mathrm{id}_V]_{\{\overline{e}_1,\overline{e}_2,\overline{e}_4\}},\ \ N\colon={}_{\{\overline{e}_1,\overline{e}_2,\overline{e}_4\}}[\mathrm{id}_V]_{\{\overline{e}_2,\overline{e}_3,\overline{e}_4\}}$$

which are inverse to each other.

The choice of which matrix to use depends on which way you wish to change coordinates: we have

$${}_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_1,\overline{e}_2,\overline{e}_4\}}={}_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_2,\overline{e}_3,\overline{e}_4\}}\cdot M,\ \ {}_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_2,\overline{e}_3,\overline{e}_4\}}={}_{\{\overline{e}'_1,\overline{e}'_2\}}[\overline{A}]_{\{\overline{e}_1,\overline{e}_2,\overline{e}_4\}}\cdot N.$$

We'll compute M, and then we'll invert it to find N (or could just as well directly compute N by the same method, and check our results are inverse to each other as a safety check). Then we'll see that our computed matrices do work as the theory ensures they must, relative to our calculations of matrices for $\overline{A}$ relative to two different choices of basis in $\mathbb{R}^4/L$. By starting at the two bases of $\mathbb{R}^4/L$ which are involved, it is clear that

$$M=\begin{pmatrix}a&1&0\\b&0&0\\c&0&1\end{pmatrix},$$

where

$$\overline{e}_1=a\overline{e}_2+b\overline{e}_3+c\overline{e}_4$$

in $\mathbb{R}^4/L$, or equivalently

$$e_1=ae_2+be_3+ce_4+dw$$

with w the indicated basis vector of L and $d\in\mathbb{F}$.

We need to explicate the fact that $\{e_2,e_3,e_4,w\}$ is a basis of $\mathbb{R}^4$ by solving the system of equations

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}d\\a-d\\b+2d\\c\end{pmatrix},$$

so $d=1$, $a=d=1$, $b=-2d=-2$, $c=0$. Thus,

$$M=\begin{pmatrix}1&1&0\\-2&0&0\\0&0&1\end{pmatrix},$$

and likewise (either by inverting or doing a similar direct calculation, or perhaps even both!) we find

$$N=\begin{pmatrix}0&-\frac{1}{2}&0\\1&\frac{1}{2}&0\\0&0&1\end{pmatrix}.$$

Now the calculation comes down to checking one of the following identities:

$$\begin{pmatrix}\frac{28}{11}&-\frac{10}{11}&\frac{56}{11}\\1&1&-2\end{pmatrix}\stackrel{?}{=}\begin{pmatrix}-\frac{10}{11}&-\frac{19}{11}&\frac{56}{11}\\1&0&-2\end{pmatrix}\begin{pmatrix}1&1&0\\-2&0&0\\0&0&1\end{pmatrix},$$

$$\begin{pmatrix}-\frac{10}{11}&-\frac{19}{11}&\frac{56}{11}\\1&0&-2\end{pmatrix}\stackrel{?}{=}\begin{pmatrix}\frac{28}{11}&-\frac{10}{11}&\frac{56}{11}\\1&1&-2\end{pmatrix}\begin{pmatrix}0&-\frac{1}{2}&0\\1&\frac{1}{2}&0\\0&0&1\end{pmatrix},$$

both of which are straightfoward to verify.

Finally, to show explicitly that $\overline{e}_1, \overline{e}_2, \overline{e}_3$ are not a basis of $\mathbb{R}^4/L$, we consider the congruence class $x = \overline{e}_4$. This vector cannot lie in the $\mathbb{R}$-span of $\overline{e}_1, \overline{e}_2, \overline{e}_3$ for otherwise there would exist $a, b, c \in \mathbb{R}$ with

$$e_4 - ae_1 - be_2 - ce_3 \in \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix},$$

but this would express e_4 in the span of four vectors which all have last coordinate zero. This is impossible.

5. The Interaction of the Quotient and an Inner Product

Let V a vector space over $\mathbb{R}$, $\dim V < \infty$, and $W \subseteq V$ a subspace. Further suppose that V is equipped with an inner product $\langle \cdot, \cdot \rangle$. Then we could identify V/W with $W^\perp$ since $W \oplus W^\perp = V$. In this case, the isomorphism is particularly easy to construct. Indeed, every element $v \in V$ can be uniquely written as $u + w$, where $u \in W^\perp$ and $w \in W$ and we identify v with u. In other words, the direct sum decomposition implies that restricting the projection $\pi \colon V \to V/W$ to $W^\perp$ induces an isomorphism of $W^\perp$ onto V/W, and by composing with the inverse of this isomorphism we get a composite map $V \to V/W \simeq W^\perp$ that is exactly the orthogonal projection onto $W^\perp$ (in view of how the decomposition of V into a direct sum of W and $W^\perp$ is made). In this sense, when given an inner product on our finite-dimensional V over $\mathbb{R}$ we may use the inner product structure to set up a natural identification of quotient spaces V/W with subspaces of the given vector space V, and we see that this orthogonal picture is exactly what was suggested in the example in Section 1 (see Figure 1). However, since this mechanism depends crucially on the inner product structure, we stress a key point: if we change the inner product then the notion of orthogonal projection (and hence the physical subspace $W^\perp$ in general) will change, so this method of linearly putting V/W back into V depends on more than just the linear structure. Consequently, if we work with auxiliary structures that do not respect the inner product (such as non-orthogonal self maps of V), then we cannot expect such data to interact well with this construction even when such data may given something well-posed on the level of abstract vector spaces.

Let us consider an eample to illustrate what can go wrong if we are too sloppy in this orthgonal projection method of putting V/W back into V. Suppose that V' is another finite-dimensional vector space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle'$, $W' \subseteq V'$ a subspace of V and $T \colon V \to V'$ a linear map with $T(W) \subseteq W'$. Then it is not true, in general, that $T(W^\perp) \subseteq (W')^\perp$. As an example, consider $V = \mathbb{R}^2$, $W = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, $V' = \mathbb{R}^3$ and $W' = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and let $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map $(x, y) \mapsto (x, y, y)$. Assume that V and V' are both equipped with the standard Euclidean inner product. Clearly, $T(W) \subseteq W'$. On the other hand, $(0, 1) \in W^\perp$ but $T((0, 1)) = (0, 1, 1) \notin (W')^\perp$ and hence $T(W^\perp) \nsubseteq (W')^\perp$. So the restriction of T to $W^\perp$ does not land in $W'^\perp$ and hence there is no "orthogonal projection" model inside of V and V' for the induced map $\overline{T}$ betwen the quotients V/W and V'/W'. (Note that T does not preserve the inner products in this example.) Hence, although the induced quotient map $\overline{T}$ makes sense here, as T carries W into W', if we try to visualize the quotients back inside of the original space via orthogonal projection then we have a hard time seeing $\overline{T}$.

6. The Quotient and the Dual

Let V be a vector space over $\mathbb{F}$, $\dim V < \infty$ and $W \subseteq V$ a subspace. Then $(V/W)^*$, the dual space of V/W, can naturally be thought of as a linear subspace of V^*. More precisely, there exists a natural isomorphism $\varphi\colon U \to (V/W)^*$, where $U = \{f \in V^* : f(W) = 0\}$. The isomorphism can be easily constructed by appealing to the universal property of the quotient. Indeed, let $f \in U$. Then $f\colon V \to \mathbb{F}$ is linear with $W \subseteq \ker(f)$ and so there exists a unique map $g\colon V/W \to \mathbb{F}$, in other words $g \in (V/W)^*$, such that $f = g \circ \pi$. Define $\varphi(f) = g$, where f and g as before. The linearity of φ is clear from the construction and the injectivity is also clear since if $\varphi(f_1) = g = \varphi(f_2)$, then $f_1 = g \circ \pi = f_2$. To see that φ is surjective, pick $g \in (V/W)^*$ arbitrary and let $f = g \circ \pi \in U$. By uniqueness, $\varphi(f) = g$ follows from the definition of φ.

Conversely, we can show that every subspace W' of V^* has the form $(V/W)^*$ for a unique quotient V/W of V. In other words, the subspaces of V^* are in bijective correspondence with the quotients of V via duality. Indeed, to give a quotient that is the "same" as to give a surjective linear map $V \to V'$, and to give a subspace of V^* is the "same" as to give an injective linear map $U^* \to V^*$, and we know that a linear map between finite-dimensional vector spaces is surjective if and only if its dual injective (and vice versa). By means of double-duality, these two procedures are the reverse of each other: if $V \twoheadrightarrow V'$ is a quotient and we form the subspace $V'^* \to V^*$, then upon dualizing again we recover (by double duality) the original surjection, which is to say that V' is the quotient by exactly the subspace of vectors in V killed by the functionals coming from V'^*. That is, the concrete converse procedure to go from subspaces of V^* to quotients of V is the following: given a subspace of V^*, we form the quotient of V modulo the subspace of vectors killed by all functionals in the given subspace of V^*. In the context of inner product spaces, where we use the inner product to indentify quotients with subspaces of the original space, this procedure translates into the passage between subspaces of V and their orthogonal complements (and the fact that the above two procedures are reverse to each other corresponds to the fact that the orthogonal complement of the orthogonal complements is the initial subspace).

7. More on the Quotient and Matrices

Let V_1 and V_2 be vector spaces over $\mathbb{F}$ of dimension n_1 and n_2, respectively, $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ subspaces of dimension m_1 and m_2, respectively, and $T\colon V_1 \to V_2$ a linear map such that $T(W_1) \subseteq W_2$. Let B_1 and B_2 be bases of V_1 and V_2, respectively, that are extensions of bases of W_1 and W_2, resectively. Let A be the matrix representation of T with respect to B_1 and B_2. Then, the lower right $(n_1 - m_1) \times (n_2 - m_2)$ block of A exactly corresponds to the matrix representation of the induced map $\overline{T}\colon V_1/W_1 \to V_2/W_2$ with respect to the bases of V_1/W_1 and V_2/W_2 coming from the non-zero vectors in the sets $\pi_1(B_1)$ and $\pi_2(B_2)$, π_1 and π_2 the usual projection maps.

Proof. Let $m = n_2 - m_2$ and $\overline{v_1}, \ldots, \overline{v_m} \in V_2/W_2$ be the non-zero vectors in $\pi_2(B_2)$. Note that these vectors form a basis of V_2/W_2. Further, let $v \in B_1$ with $v \notin W_1$. Then

$$\begin{aligned}\overline{T(v)} &= \overline{T}(\overline{v}) \\ &= \alpha_1 \overline{v_1} + \cdots + \alpha_m \overline{v_m}.\end{aligned}$$

Hence there exists $w \in W_2$ such that $T(v) = \alpha_1 v_1 + \cdots + \alpha_m v_m + w$. The claim follows since the matrix representation of a linear map with respect to choices of ordered bases arises exactly as the coefficients of the images of vectors making up the ordered basis of the source vector space written with respect ordered basis of the target vector space basis. □

Now, let $V_1 = V_2 = V$ and $W_1 = W_2 = W$ and $T\colon V \to V$ a linear map with $T(W) \subseteq W$. Let $T|W\colon W \to W$ be the self map induced by T, and let $\overline{T}\colon V/W \to V/W$ be the map induced by V/W. We shall now apply the above result to show that $\det(T) = \det(T|W)\det(\overline{T})$. To this end, one only needs to note that the upper left block M_{11} of the matrix representation

$$A = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

of T with respect to a basis B of V extended from a basis B' of W will correspond to the matrix representation of $T|W$ with respect to the basis B' and also that the entries of the lower left block M_{21} of A will be 0. Both of these facts are immediate since the restriction $T|W$ maps into W. Furthermore, the previous result implies that M_{22} exactly corresponds to the matrix representation of $\bar{T}$ with respect to the basis B'.

As another application, consider the following. Let V be a vector space of finite dimension over $\mathbb{C}$ and $T\colon V \to V$ a linear map. We will use the idea of the quotient space to show that there exists a basis B of V such that the map T written as a matrix with respect to the basis B is upper triangular.

Proof. The proof is by induction on the dimension $n \in \mathbb{N}_0$. The case where $n = 0$, i.e. $V = 0$, is somewhat meaningless and so we ignore it. The case where $n = 1$ is also trivial. Now suppose that for some fixed $n \in \mathbb{N}$ that for every vector space V over $\mathbb{C}$ of dimension n that every linear map $T\colon V \to V$ can be expressed as an upper triangular matrix with respect to some ordered basis B of V. Let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$ and $S\colon W \to W$ a linear map. Recall that by the fundemental theorem of algebra $\mathbb{C}$ is algebraically closed, i.e. every polynomial has a root. In particular, it is immediate from this fact that there exists an eigenvector $w \in W$ of S. Let $L = \operatorname{span}\{w\}$. Note that $S(L) \subseteq L$ since w is an eigenvector of S and further that $\dim W/L = n$. By the induction hypothesis, there exists an ordered basis $B = \{\overline{w_1}, \ldots, \overline{w_n}\}$ of W/L such that induced map $\overline{S}\colon W/L \to W/L$ expressed as a matrix with respect to B is upper triangular. Let $B' = \{w, w_1, \ldots, w_n\}$. Then B' is an ordered basis of W (recall Section 2). Furthermore, S written as a matrix A with respect to B' is upper triangular.

$$A = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Above we express A in block form. M_{11} is a 1×1 matrix, M_{12} a $1 \times n$ matrix, M_{21} an $n \times 1$ matrix and M_{22} is an $n \times n$ matrix. Since w is an eigenvector, it is clear that M_{11} corresponds to the eigenvalue associated with w and that M_{21} consists only of zeroes. M_{22} exactly agrees with the matrix expression of $\overline{S}$ with respect to B (recall the construction of the induced map) and so is upper triangular. For our purposes, the form of M_{12} is irrelevant and thus A is upper triangular. $\square$