Linear Algebra

Final Test

2008.1.15.

1. (12%) Let V, W be vector spaces, $T : V \to W$ be linear, and let $U : V \to W$ be a function. Give the definitions of the followings.
   (a) (3%) U is linear.
   (b) (3%) $N(T)$.
   (c) (3%) $R(T)$.
   (d) (3%) $V \cong W$ (V is isomorphic to W).

2. (40%) Determine (by proof or counterexample) the truth or falsity of the following statements. (Note: you need to explain why the example you give is a counterexample if the statement is false.)
   (a) (5%) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.
   (b) (5%) Let V, W be vector spaces, if $T : V \to W$, $U : V \to W$ are two linear transformations such that $N(T) = N(U)$, $R(T) = R(U)$, then $T = U$.
   (c) (5%) There does not exist linear transformation $T : V \to W$ such that $N(T) = R(T)$.
   (d) (5%) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear and $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$, then $T(8, 11) = (5, -3, 16)$.
   (e) (5%) Let $A \in M_{n\times n}(\mathbb{R})$, if $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.
   (f) (5%) Let V be a vector space and $T : V \to W$ be linear, then $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$ (T_0 is the zero transformation from V to V).
   (g) (5%) Let $A \in M_{n\times n}(\mathbb{R})$, if $A^2 = O$, then $A = O$ (O is the zero matrix).
   (h) (5%) Let $A, B \in M_{n\times n}(R)$, if $AB = O$, then $BA = O$ (O is the zero matrix).

3. (30%) Let V, W be vector spaces and $T : V \to W$ be linear.
   (a) (5%) Prove that $N(T)$ and $R(T)$ are subspaces of V and W, respectively.
   (b) (5%) Prove that if $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V, then

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \cdots, T(v_n)\}).$$

(c) (10%) Prove that if V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

(d) (5%) Prove that T is one-to-one if and only if $N(T) = \{0\}$.

(e) (5%) Prove that if $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V, then for $w_1, w_2, \cdots, w_n \in W$, there exists exactly one linear transformation $U : V \to W$ such that $U(v_i) = w_i$ for $i = 1, 2, \cdots n$.

4. (10%)

(a) (6%) Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(f) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix},$$

find $N(T)$, $R(T)$, nullity(T) and rank(T).

(b) (4%) Let $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be a linear transformation defined by $T(A) = \text{tr}(A)$, find nullity(T) and rank(T) (tr(A) is the trace of A).

5. (25%)

(a) (8%) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$, β be the standard ordered basis for $\mathbb{R}^2$, $\alpha = \{(1, 2), (2, 3)\}$, and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Find $[T]_\beta^\gamma$ and $[T]_\alpha^\gamma$.

(b) (5%) Let $\beta = \{1, x, x^2\}$ and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the ordered bases for $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$, respectively, and $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by $T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Compute $[T]_\beta^\gamma$.

(c) (7%) Let $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the ordered bases for $M_{2\times 2}(\mathbb{R})$, and $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by $T(A) = A^t$. Find $[T]_\gamma$ and $[T(B)]_\gamma$, where $B = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

(d) (5%) Let $g(x) = 3 + x$, $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be linear transformations defined by $T(f(x)) = f'(x)g(x) + 2f(x)$ and $U(a+bx+cx^2) = (a+b, c, a-b)$, respectively, and let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. Find $[UT]_\beta^\gamma$.

6. (15%) Let V, W be vector spaces over F and $T : V \to W$ be linear.

(a) (5%) Prove that if $U : V \to W$ is linear and $a \in F$, then $aT + U$ is linear.

(b) (5%) Prove that if $U : W \to Z$ is linear, then $UT : V \to Z$ is linear.

(c) (5%) Prove that if T is invertible, then $T^{-1} : W \to V$ is linear.

7. (15%)

   (a) (5%) Let V be a finite-dimensional vector space having ordered basis β, and $u_1, u_2, \cdots, u_n \in V$. Prove that $[u_1 + u_2 + \cdots + u_n]_\beta = [u_1]_\beta + [u_2]_\beta + \cdots + [u_n]_\beta$. (Hint: First show that $[u_1 + u_2]_\beta = [u_1]_\beta + [u_2]_\beta$).

   (b) (10%) Let V, W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $T : V \to W$ be linear. Prove that for each $u \in V$, we have $[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.

8. (15%)

   (a) (5%) Let V and W be finite-dimensional vector spaces over F, and let $T : V \to W$ be linear. Prove that if T is invertible, then $\dim(V) = \dim(W)$.

   (b) (5%) Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Prove that the following conditions are equivalent.

      i. T is invertible

      ii. T is one-to-one

      iii. T is onto

   (c) (5%) Let V and W be finite-dimensional vector spaces over F. Prove that V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

9. (10%)

   (a) (4%) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Show that T is invertible.

   (b) (6%) Let

   $$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

   Construct an isomorphism from V to $\mathbb{R}^3$. (You need to show that the function you construct is indeed an isomorphism).

10. (30%)

   (a) (5%) Let V, W be vector spaces, and $T : V \to W$ be linear. Suppose that T is one-to-one and S is a subset of V. Prove that if S is linearly independent, then $T(S)$ is linearly independent.

   (b) (5%) Let V, W be vector spaces, S be a subset of V, and $T : V \to W$ be linear. Prove that if $T(S)$ is linearly independent, then S is linearly independent.

(c) (6%) Let V, W be finite-dimensional vector spaces, β be an ordered basis for V, and $T : V \to W$ be linear. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

(d) (7%) Let V, W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Prove that for any subspace V_0 of V, $T(V_0)$ is a subspace of W, and $\dim(V_0) = \dim(T(V_0))$.

(e) (7%) Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W, and let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$.