MATH 413-513

RECENT RESULTS

1. Results on Linear Transformations

Here is a list of the important results we have proven about linear transformations between vector spaces.

Theorem 1. *Let V and W be vector spaces over a field $\mathbb{F}$. Let $\{v_1, v_2, \cdots, v_n\} \subset V$ be a basis of V. Let $\{w_1, w_2, \cdots w_n\} \subset W$ be any collection of vectors. There is a unique linear map $T : V \to W$ with*

$$T(v_i) = w_i \quad \textit{for all } 1 \le i \le n\,.$$

Proposition 1. *Let V and W be vector spaces over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, W)$. Then,*
i) $T(0) = 0$.
ii) $\operatorname{null}(T) \subset V$ *is a subspace of* V.
ii) $\operatorname{range}(T) \subset W$ *is a subspace of* W.

Proposition 2. *Let V and W be vector spaces over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if* $\operatorname{null}(T) = \{0\}$.

Theorem 2. *Let V and W be vector spaces over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, W)$. If V is finite dimensional, then* $\operatorname{range}(T)$ *is finite dimensional and*

$$\dim(V) = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T))$$

Corollary 1. *Let V and W be vector spaces over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, W)$. If V is finite dimensional, then*
i) If $\dim(V) > \dim(W)$, *then T is not injective.*
ii) If $\dim(V) < \dim(W)$, *then T is not surjective.*

Proposition 3. *Let U, V, and W be finite dimensional vector spaces over a field $\mathbb{F}$. Fix bases in U, V, and W. For any $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$,*

$$M(TS) = M(T)M(S)\,.$$

Proposition 4. *Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$. Fix bases in V and W. For any $T \in \mathcal{L}(V, W)$ and any $v \in V$,*

$$M(Tv) = M(T)M(v)\,.$$

Proposition 5. *Let V and W be vector spaces over a field $\mathbb{F}$. A linear mapping $T \in \mathcal{L}(V, W)$ is invertible if and only if T is injective and surjective.*

Theorem 3. *Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$. V and W are isomorphic if and only if*

$$\dim(V) = \dim(W).$$

Theorem 4. *Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, V)$.*
The following are equivalent:
i) T is invertible.
ii) T is injective.
iii) T is surjective.

Proposition 6. *Let V be a finite dimensional vector space over $\mathbb{F}$ and let $T \in \mathcal{L}(V, V)$. $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $T - \lambda I$ is not invertible. (By the previous result, you can replace* **invertible** *with either* **injective** *or* **surjective***.)*

Theorem 5. *Let V be a vector space over $\mathbb{F}$, $T \in \mathcal{L}(V, V)$, and let $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{F}$ be distinct eigenvalues of T with corresponding eigenvectors $v_1, v_2, \cdots, v_n \in V$. The set $\{v_1, v_2, \cdots, v_n\} \subset V$ is linearly independent.*

Corollary 2. *Let V be a finite dimensional vector space over $\mathbb{F}$. Any $T \in \mathcal{L}(V, V)$ has at most $\dim(V)$ distinct eigenvalues.*

Proposition 7. *Let V be a finite dimensional vector space over $\mathbb{F}$ and let $T \in \mathcal{L}(V, V)$ have $\dim(V)$ distinct eigenvalues. Then V has a basis consisting of eigenvectors of T and with respect to this basis (on V as the domain of T and on V as the range of T) the matrix $M(T)$ is diagonal.*

Theorem 6. *Let $V \neq \{0\}$ be a finite dimensional vector space over $\mathbb{C}$. Then each $T \in \mathcal{L}(V, V)$ has at least one eigenvalue.*

Proposition 8. *Let V be a finite dimensional vector space over $\mathbb{F}$. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis in V and take $T \in \mathcal{L}(V, V)$.*
The following are equivalent:
i) The matrix $M(T)$ with respect to $\{v_1, v_2, \cdots, v_n\}$ is upper diagonal.
ii) For each $k = 1, 2, \cdots, n$, $T(v_k) \in \operatorname{span}(v_1, v_2, \cdots, v_k)$.
iii) For each $k = 1, 2, \cdots, n$, the subspace $U_k = \operatorname{span}(v_1, v_2, \cdots, v_k)$ is a T invariant subspace of V.

Theorem 7. *Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V, V)$. Then, there is a basis of V in which $M(T)$ is upper diagonal.*

Proposition 9. *Let V be a finite dimensional vector space over $\mathbb{F}$. Let $T \in \mathcal{L}(V, V)$ and suppose there is a basis in V in which $M(T)$ is upper diagonal. Then*
i) T is invertible if and only if all entries on the diagonal of $M(T)$ are nonzero.
ii) The eigenvalues of T are precisely the diagonal entries of $M(T)$.