# Homework - S403

Due September 24, 2013

**Please write on only one side of the sheet.**

1. Let $G$ be a group.

(a) Prove that if $G$ has exactly three subgroups, then $G$ is finite cyclic and $|G|$ is $p^2$ for some prime $p$.

(b) Prove that if $G$ has exactly four subgroups, then $G$ is finite cyclic and $|G|$ is either $p^3$ for some prime $p$ or $pq$ for distinct primes $p$ and $q$.

2. Let $G$ be a (possibly infinite) group. Let $H$ be a subgroup.

(a) Prove that $\tilde{H} = \cap_{g \in G} gHg^{-1}$ is a normal subgroup of $G$. Then prove that $\tilde{H}$ is the largest normal subgroup of $G$ contained in $H$, in the sense that if $K$ is any normal subgroup of $G$ contained in $H$, then $K \subseteq \tilde{H}$.

(b) Now suppose $G$ contains a subgroup of finite index. Prove that $G$ contains a normal subgroup of finite index.

3. Let $G$ be a group and let $H$ be a subgroup. Prove that if $H$ has index two in $G$, then $H$ is normal.

4. Find all the subgroups of $C_{12}$, the cyclic group of order 12.

5. In $D_4$, let $N = < R_2 >$, the subgroup generated by $R_2$. We have seen that $N$ is normal. The quotient group $D_4/N$ is a group you know. What is it?

6. Let $f : G_1 \to G_2$ be a group homomorphism and let $N$ denote the kernel of $f$. In class we showed there is an induced homomorphism from $G_1/N$ to $G_2$. Generalize this by showing that if $K$ is a normal subgroup of $G_1$ such that $K \subseteq N$, then there is an induced homomorphism from $G_1/K$ to $G_2$.