Homework - S403

Due September 10, 2013

Please write on only one side of the sheet.

Note: $\mathbf{Z}$ denotes the set of integers and $\mathbf{R}$ denotes the set of real numbers

1. (a) Let S, T be sets and let $f : S \to T$ be a function. Define a relation $\sim$ on S by $s_1 \sim s_2$ if $f(s_1) = f(s_2)$. Prove that $\sim$ is an equivalence relation.

(b) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = x^2 - y^2$. Draw the equivalence classes for the equivalence relation determined (as in part (a)) by f.

2. Determine all subgroups of D_4.

3. Let $m, n \in \mathbf{Z}$. We have proved there is a unique positive integer l such that $m\mathbf{Z} \cap n\mathbf{Z} = l\mathbf{Z}$. Prove that l is the least common multiple of m and n.

4. Let H and K be subgroups of a group G. Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

5. Let S be a set and let $f : S \to S$ be a one-to-one, onto function. Then f has an inverse and If k is an integer it makes sense to talk about f^k (So $f^2(s) = f(f(s))$ and $f^{-2}(s) = f^{-1}(f^{-1}(s))$ and so on). Define a relation on S by $s_1 \sim s_2$ if there is an integer k such that $f^k(s_1) = s_2$. Show this is an equivalence relation. The equivalence classes for this relation are called the <u>orbits</u> of the function f.

6. (a) Let H be a subgroup of a group G. Define a relation $\sim$ on G by $g_1 \sim g_2$ if $g_1 g_2^{-1} \in H$. Prove that this is an equivalence relation and that the equivalence classes are precisely the "right" cosets Hg for $g \in G$.

(b) Give an example of a subgroup H of a group G in which there are elements $g_1, g_2 \in G$ such that $g_1 H = g_2 H$, but $Hg_1 \neq Hg_2$.

(c) Let H be a subgroup of a group G and let $g_1, g_2 \in G$. Prove that $g_1 H = g_2 H$ if and only if $Hg_1^{-1} = Hg_2^{-1}$.

7. Let G be a group in which for all $g \in G$, $g^2 = e$. Prove G is abelian.