M343 Homework 6

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Section 3.5

2. $y'' + 2y' + 5y = 3sin(2t)$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 + 2r + 5 = 0 \iff r = -1 \pm 2i$. The solution in this case is:

$$\boxed{y_h = C_1 e^{-t} cos(2t) + C_2 e^{-t} sin(2t)}$$

$\underline{y_p}$: Guess $y_p = Acos(2t) + Bsin(2t)$. Then, $y_p' = -2Asin(2t) + 2Bcos(2t)$ and $y_p'' = -4Acos(2t) - 4Bsin(2t)$. y_p must satisfy the differential equation:

$$\begin{aligned} 3sin(2t) &= y_p'' + 2y_p' + 5y_p \\ &= -4Acos(2t) - 4Bsin(2t) + 2(-2Asin(2t) + 2Bcos(2t)) + 5(Acos(2t) + Bsin(2t)) \\ &= (A + 4B)cos(2t) + (-4A + B)sin(2t) \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} A + 4B = 0 \Longrightarrow A = -4B \Longrightarrow \boxed{A = -\frac{12}{17}} \\ \\ -4B + B = 3 \Longrightarrow 17B = 3 \Longrightarrow \boxed{B = \frac{3}{17}} \end{cases}$$

The general solution is:

$$\boxed{y = C_1 e^{-t} cos(2t) + C_2 e^{-t} sin(2t) - \frac{12}{17} cos(2t) + \frac{3}{17} sin(2t)}$$

3. $y'' - 2y' - 3y = -3te^{-t}$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 - 2r - 3 = 0 = 0 \iff r_1 = 3;\ r_2 = -1$. The solution in this case is:

$$\boxed{y_h = C_1 e^{3t} + C_2 e^{-t}}$$

$\underline{y_p}$: Guess $y_p = t(At + B)e^{-t} \iff y_p = e^{-t}(At^2 + Bt)$. Then:

$$y_p' = e^{-t}[-At^2 + (2A - B)t + B]$$

$$y_p'' = e^{-t}[At^2 + (-4A + B)t + 2A - 2B]$$

y_p must satisfy the differential equation:

$$\begin{aligned} -3te^{-t} &= y_p'' - 2y_p' - 3y_p \\ &= e^{-t}[At^2 + (-4A + B)t + 2A - 2B] - 2(e^{-t}[-At^2 + (2A - B)t + B]) - 3(e^{-t}(At^2 + Bt)) \\ &= e^{-t}[t^2(A + 2A - 3A) + t(-4A + B - 4A + 2B - 3B) + (2A - 2B - 2B)] \\ &= e^{-t}[t(-8A) + 2A - 4B] \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} -8A = -3 \Longrightarrow \boxed{A = \dfrac{3}{8}} \\ \\ 2A - 4B = 0 \Longrightarrow A = 2B \Longrightarrow \boxed{B = \dfrac{3}{16}} \end{cases}$$

The general solution is:

$$\boxed{y = C_1e^{3t} + C_2e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}}$$

11. $y'' + y' + 4y = e^t - e^{-t}$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 + r + 4 = 0 \iff r = \dfrac{-1 \pm \sqrt{15}i}{2}$. The solution in this case is:

$$\boxed{y_h = C_1e^{-t/2}cos\left(\frac{\sqrt{15}}{2}t\right) + C_2e^{-t/2}sin\left(\frac{\sqrt{15}}{2}t\right)}$$

$\underline{y_p}$: Guess $y_p = Ae^t + Be^{-t}$. Then, $y_p' = Ae^t - Be^{-t}$ and $y_p'' = Ae^t + Be^{-t}$.
y_p must satisfy the differential equation:

$$\begin{aligned} e^t - e^{-t} &= y_p'' + y_p' + 4y_p \\ &= Ae^t + Be^{-t} + Ae^t - Be^{-t} + 4Ae^t + 4Be^{-t} \\ &= 6Ae^t + 4Be^{-t} \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} 6A = 1 \Longrightarrow \boxed{A = \dfrac{1}{6}} \\ \\ 4B = -1 \Longrightarrow \boxed{B = -\dfrac{1}{4}} \end{cases}$$

The general solution is:

$$\boxed{y = C_1e^{-t/2}cos\left(\frac{\sqrt{15}}{2}t\right) + C_2e^{-t/2}sin\left(\frac{\sqrt{15}}{2}t\right) + \frac{e^t}{6} - \frac{e^{-t}}{4}}$$

13. $y'' + y' - 2y = 2t$, $y(0) = 0$; $y'(0) = 1$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 + r - 2 = 0 = 0 \iff r_1 = -2$; $r_2 = 1$. The solution in this case is:

$$\boxed{y_h = C_1e^{-2t} + C_2e^t}$$

$\underline{y_p}$: Guess $y_p = At + B$. Then, $y_p' = A$ and $y_p'' = 0$.

y_p must satisfy the differential equation:

$$\begin{aligned} 2t &= y'' + y' - 2y \\ &= 0 + A - 2(At + B) \\ &= -2At + A - 2B \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} -2A = 2 \Longrightarrow \boxed{A = -1} \\ \\ A - 2B = 0 \Longrightarrow -1 - 2B = 0 \Longrightarrow \boxed{B = -\dfrac{1}{2}} \end{cases}$$

The general solution is:

$$\boxed{y = C_1 e^{-2t} + C_2 e^t - t - \frac{1}{2}}$$

Now, solve for C_1, C_2:

$$\begin{cases} y(0) = 0 = C_1 + C_2 - \dfrac{1}{2} \Longrightarrow C_1 + C_2 = \dfrac{1}{2} \Longrightarrow \boxed{C_1 = -\dfrac{1}{2}} \\ \\ y'(0) = 1 = -2C_1 + C_2 - 1 \Longrightarrow -2C_1 + C_2 = 2 \Longrightarrow \boxed{C_2 = 1} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = -\frac{1}{2} e^{-2t} + e^t - t - \frac{1}{2}}$$

15. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1; \; y'(0) = 1.$ The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 - 2r + 1 = 0 = 0 \iff (r-1)^2 = 0$. The solution in this case is:

$$\boxed{y_h = C_1 e^t + C_2 t e^t}$$

$\underline{y_p}$: Guess $y_p = t^2(At + B)e^t + C \iff y_p = (At^3 + Bt^2)e^t + C$. Then:

$$y_p' = e^t[At^3 + (3A + B)t^2 + 2Bt]$$
$$y_p'' = e^t[At^3 + (6A + B)t^2 + (6A + 4B)t + 2B]$$

y_p must satisfy the differential equation:

$$\begin{aligned} te^t + 4 &= y_p'' - 2y_p' + y_p \\ &= e^t[At^3 + (6A + B)t^2 + (6A + 4B)t + 2B] - 2(e^t[At^3 + (3A + B)t^2 + 2Bt]) + (At^3 + Bt^2)e^t + C \\ &= e^t[6At + 2B] + C \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} 6A = 1 \Longrightarrow \boxed{A = \dfrac{1}{6}} \\ \\ 2B = 0 \Longrightarrow \boxed{B = 0} \\ \boxed{C = 4} \end{cases}$$

The general solution is:

$$\boxed{y = C_1 e^t + C_2 t e^t + \frac{1}{6} t^3 e^t + 4}$$

Now, solve for C_1, C_2:

$$\begin{cases} y(0) = 1 = C_1 + 4 \Longrightarrow \boxed{C_1 = -3} \\ \\ y'(0) = 1 = -3 + C_2 \Longrightarrow \boxed{C_2 = 4} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = -3e^t + 4te^t + \frac{1}{6} t^3 e^t + 4}$$

17. $y'' + 4y = 3sin(2t)$, $y(0) = 2$; $y'(0) = -1$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 + 4 = 0 = 0 \iff r = \pm 2i$. The solution in this case is:

$$\boxed{y_h = C_1 cos(2t) + C_2 sin(2t)}$$

$\underline{y_p}$: Guess $y_p = t[Asin(2t) + Bcos(2t)]$. Then:

$$y_p' = [Asin(2t) + Bcos(2t)] + t[2Acos(2t) - 2Bsin(2t)]$$
$$y_p'' = 4Acos(2t) - 4Bsin(2t) - 4tAsin(2t) - 4Btcos(2t)$$

y_p must satisfy the differential equation:

$$\begin{aligned} 3sin(2t) &= y_p'' + 4y_p \\ &= 4Acos(2t) - 4Bsin(2t) - 4tAsin(2t) - 4Btcos(2t) + 4t[Asin(2t) + Bcos(2t)] \\ &= 4Acos(2t) - 4Bsin(2t) - 4tAsin(2t) - 4Btcos(2t) + 4Atsin(2t) + 4Btcos(2t) \\ &= 4Acos(2t) - 4Bsin(2t) \end{aligned}$$

From which we can setup the following system of linear equations:

$$\begin{cases} 4A = 0 \Longrightarrow \boxed{A = 0} \\ -4B = 3 \Longrightarrow \boxed{B = -\frac{3}{4}} \end{cases}$$

The general solution is:

$$\boxed{y = C_1 cos(2t) + C_2 sin(2t) - \frac{3}{4} tcos(2t)}$$

Now, solve for C_1, C_2:

$$\begin{cases} y(0) = 2 = C_1 \Longrightarrow \boxed{C_1 = 2} \\ y'(0) = -1 = -3 + C_2 \Longrightarrow 2C_2 = \frac{3}{4} - 1 \Longrightarrow \boxed{C_2 = -\frac{1}{8}} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = 2cos(2t) - \frac{1}{8} sin(2t) - \frac{3}{4} tcos(2t)}$$

18. $y'' + 2y' + 5y = 4e^{-t}cos(2t)$, $y(0) = 1$; $y'(0) = 0$. The general solution is given by:

$y_g = y_h + y_p$, y_p is the solution to the associated homogeneous equation and y_p is the particular solution

$\underline{y_h}$: Characteristic equation: $r^2 + 2r + 5 = 0 = 0 \iff r = -1 \pm 2i$. The solution in this case is:

$$\boxed{y_h = C_1 e^{-t} cos(2t) + C_2 e^{-t} sin(2t)}$$

$\underline{y_p}$: Guess $y_p = t[Asin(2t) + Bcos(2t)]e^{-t}$. Then:

$$y_p' = (e^{-t} - te^{-t})[Asin(2t) + Bcos(2t)] + te^{-t}[2Acos(2t) - 2Bsin(2t)]$$

$$y_p'' = (-2e^{-t} + te^{-t})[Asin(2t) + Bcos(2t)] + 2(e^{-t} - te^{-t})[2Acos(2t) - 2Bsin(2t)] + te^{-t}[-4Asin(2t) - 4Bcos(2t)]$$

y_p must satisfy the differential equation:

$$4e^{-t}cos(2t) \quad = \quad y_p'' + 2y_p' + 5y_p$$

From which we can setup the following system of linear equations:

$$\begin{cases} -2Bt - 4B = 0 \Longrightarrow \boxed{B = 0} \\ 2At + 4A = 4 \Longrightarrow \boxed{A = 1} \end{cases}$$

The general solution is:

$$\boxed{y = C_1e^{-t}cos(2t) + C_2e^{-t}sin(2t) + te^{-t}sin(2t)}$$

Now, solve for C_1, C_2:

$$\begin{cases} y(0) = 1 = C_1 \Longrightarrow \boxed{C_1 = 1} \\ y'(0) = 0 = -1 + 2C_2 \boxed{C_2 = \frac{1}{2}} \end{cases}$$

The solution to the I.V.P is:

$$\boxed{y = e^{-t}cos(2t) + \frac{1}{2}e^{-t}sin(2t) + te^{-t}sin(2t)}$$

Section 3.6

4. $4y'' - 4y' + y = 16e^{t/2} \iff y'' - y' + \frac{1}{4}y = 4e^{t/2}$.

The solution to the homogeneous equation is: $r^2 - r + \frac{1}{4} = 0 \iff (r - \frac{1}{2})^2 = 0$

$$y_h = C_1e^{t/2} + C_2te^{t/2}$$

Let $y_1 = e^{t/2}$ and $y_2 = te^{t/2}$. On the one hand, using the method of variation of parameters:

$$W(y_1, y_2) = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{e^{t/2}}{2} & e^{t/2} + \frac{te^{t/2}}{2} \end{vmatrix} = e^t$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{(-te^{t/2})(4e^{t/2})}{e^t} = -4t \Longrightarrow u_1 = -2t^2 + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{(e^{t/2})(4e^{t/2})}{e^t} = 4 \Longrightarrow u_2 = 4t + C_2$$

So the solution is given by:

$$y = u_1y_1 + u_2y_2 = (-2t^2 + C_1)e^{t/2} + (4t + C_2)e^{t/2}t$$

General solution:

$$\boxed{y = C_1e^{t/2} + C_2te^{t/2} + 2t^2e^{t/2}}$$

The particular solution, using the method of variation of parameters is $\boxed{y_p = 2t^2e^{t/2}}$.

On the other hand, using the method of undetermined coefficients: guess the solution y_p :

$$y_p = At^2e^{t/2}; \Longrightarrow y_p' = 2Ate^{t/2} + \frac{A}{2}t^2e^{t/2}; \Longrightarrow y_p'' = 2Ae^{t/2} + 2Ate^{t/2} + \frac{A}{4}t^2e^{t/2}$$

The particular solution y_p must satisfy the equation:

$$\begin{aligned}
4e^{t/2} &= y_p'' - y_p' + \frac{1}{4}y_p \\
&= 2Ae^{t/2} + 2Ate^{t/2} + \frac{A}{4}t^2e^{t/2} - 2Ate^{t/2} - \frac{A}{2}t^2e^{t/2} + \frac{A}{4}t^2e^{t/2} \\
&= e^{t/2}[2A + 2At + \frac{A}{4}t^2 - 2At - \frac{A}{2}t^2 + \frac{A}{4}t^2] \\
&= e^{t/2}[t^2(\frac{A}{4} + \frac{A}{4} - \frac{A}{2}) + t(2A - 2A) + 2A]
\end{aligned}$$

From which we can conclude that $2A = 4 \iff \boxed{A = 2}$. So the particular solution is $\boxed{y_p = 2t^2e^{t/2}}$

The two methods agree.

5. $y'' + y = tan(t), \quad 0 < t < \pi/2.$

The solution to the homogeneous equation is: $r^2 + 1 = 0 \iff r = \pm i$

$$y_h = C_1cos(t) + C_2sin(t)$$

Let $y_1 = cos(t)$ and $y_2 = sin(t)$. Using the method of variation of parameters:

$$W(y_1, y_2) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix} = 1$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{-sin(t)tan(t)}{1} = \frac{-sin^2(t)}{cos(t)} \Longrightarrow u_1 = \int \frac{-sin^2(t)}{cos(t)}dt = \int cos(t) - sec(t)dt = sin(t) - ln(sec(t) + tan(t)) + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{cos(t)tan(t)}{1} = sin(t) \Longrightarrow u_2 = -cos(t) + C_2$$

So the solution is given by:

$$y = u_1y_1 + u_2y_2 = (sin(t) - ln(sec(t) + tan(t)) + C_1)cos(t) + (-cos(t) + C_2)sin(t)$$

General solution:

$$\boxed{y = C_1cos(t) + C_2sin(t) - ln(sec(t) + tan(t))cos(t)}$$

8. $y'' + 4y = 3csc(2t), \quad 0 < t < \pi/2.$

The solution to the homogeneous equation is: $r^2 + 4 = 0 \iff r = \pm 2i$

$$y_h = C_1cos(2t) + C_2sin(2t)$$

Let $y_1 = cos(t)$ and $y_2 = sin(t)$. Using the method of variation of parameters:

$$W(y_1, y_2) = \begin{vmatrix} cos(2t) & sin(2t) \\ -2sin(2t) & 2cos(2t) \end{vmatrix} = 2$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{-sin(2t)3csc(2t)}{2} = -\frac{3}{2} \Longrightarrow u_1 = -\frac{3}{2}t + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{cos(2t)3csc(2t)}{2} = \frac{3}{2}cotan(2t) \Longrightarrow u_2 = \frac{3}{4}ln(sin(2t)) + C_2$$

So the solution is given by:

$$y = u_1y_1 + u_2y_2 = (-\frac{3}{2}t + C_1)cos(2t) + (\frac{3}{4}ln(sin(2t)) + C_2)sin(2t)$$

General solution:

$$\boxed{y = C_1cos(2t) + C_2sin(2t) - \frac{3}{2}tcos(2t) + \frac{3}{4}ln(sin(2t))sin(2t)}$$

10. $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

The solution to the homogeneous equation is: $r^2 - 2r + 1 = 0 \iff (r-1)^2 = 0$

$$y_h = C_1 e^t + C_2 t e^t$$

Let $y_1 = e^t$ and $y_2 = te^t$. Using the method of variation of parameters:

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^2t + te^t - te^t = e^2t$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{-te^t \dfrac{e^t}{1+t^2}}{e^2t} = -\frac{t}{1+t^2} \Longrightarrow u_1 = -\frac{1}{2}ln(1+t^2) + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{e^t \dfrac{e^t}{1+t^2}}{e^{2t}} = \frac{1}{1+t^2} \Longrightarrow u_2 = tan^{-1}(t) + C_2$$

So the solution is given by:

$$y = u_1 y_1 + u_2 y_2 = (-\frac{1}{2}ln(1+t^2) + C_1)e^t + (tan^{-1}(t) + C_2)te^t$$

General solution:

$$\boxed{y = C_1 e^t + C_2 te^t - \frac{e^t ln(1+t^2)}{2} + te^t tan^{-1}(t)}$$

13. $t^2 y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2; \quad y_2(t) = t^{-1}.$

Before proceeding to solve this by variation of parameters, we need to write this equation in standard form:

$$y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2}$$

y_1 and y_2 are solutions of the corresponding homogeneous equation since:

$$y_1'' - \frac{2}{t^2}y_1 = 2 - 2 = 0$$

$$y_2'' - \frac{2}{t^2}y_2 = \frac{2}{t^3} - \frac{2}{t^3} = 0$$

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{-t^{-1}(3 - \frac{1}{t^2})}{-3} = \frac{1}{t} - \frac{1}{3t^3} \Longrightarrow u_1 = ln(t) + \frac{1}{6t^2} + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{t^2(3 - \frac{1}{t^2}}{-3} = -t^2 + \frac{1}{3} \Longrightarrow u_2 = -\frac{t^3}{3} + \frac{t}{3} + C_2$$

So the solution is given by:

$$y = u_1 y_1 + u_2 y_2 = (ln(t) + \frac{1}{6t^2} + C_1)t^2 + (-\frac{t^3}{3} + \frac{t}{3} + C_2)t^{-1}$$

Incorporating the quadratic term into the solution of the homogeneous equation:

$$y = C_1 t^2 + C_2 t^{-1} + t^2 ln(t) + \frac{1}{2}$$

So the particular solution is

$$\boxed{y_p = t^2 ln(t) + \frac{1}{2}}$$

17. $x^2y'' - 3xy' + 4y = x^2ln(x), \quad x > 0; \quad y_1(x) = x^2; \quad y_2(x) = x^2ln(x).$

Before proceeding to solve this by variation of parameters, we need to write this equation in standard form:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = ln(x)$$

y_1 and y_2 are solutions of the corresponding homogeneous equation since:

$$y_1'' - \frac{3}{x}y_1' + \frac{4}{x^2}y_1 = 2 - \frac{3}{x}2x + \frac{4}{x^2}x^2 = 6 - 6 = 0$$

$$y_2'' - \frac{3}{x}y_2' + \frac{4}{x^2}y_2 = 2ln(x) + 3 - \frac{3}{x}(2xln(x) + x) + \frac{4}{x^2}x^2ln(x) = 0$$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2ln(x) \\ 2x & 2xln(x) + x \end{vmatrix} = 2x^3ln(x) + x^3 - 2x^3ln(x) = x^3$$

$$u_1' = \frac{-y_2 \cdot g}{W} = \frac{-x^2ln^2(x)}{x^3} = \frac{-ln^2(x)}{x} \Longrightarrow u_1 = -\frac{ln^3(x)}{3} + C_1$$

$$u_2' = \frac{y_1 \cdot g}{W} = \frac{x^2ln(x)}{x^3} = \frac{ln(x)}{x} \Longrightarrow u_2 = \frac{ln^2(x)}{2} + C_2$$

So the solution is given by:

$$y = u_1y_1 + u_2y_2 = (-\frac{ln^3(x)}{3} + C_1)(x^2) + (\frac{ln^2(x)}{2} + C_2)x^2ln(x) = C_1x^2 + C_2x^2ln(x) + \frac{1}{6}x^2ln^3(x)$$

So the particular solution is

$$\boxed{y_p = \frac{1}{6}x^2ln^3(x)}$$

Section 4.1

4. Consider the equation: $y''' + ty'' + t^2y' + t^3y = ln(t)$. Since the functions: $t, t^2, t^3, ln(t)$ are continuous everywhere when $t > 0$, the interval in which the solution is certain to exists is $t > 0$.

7. Let $f_1t(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$. These functions are linearly dependent since:

$$k_1f_1(t) + k_2f_2(t) + k_3f_3(t) = 0$$

Hold for every t, in particular $t = -1, 0, 1$

$$\begin{cases} -5k_1 + 2k_2 + 3k_3 & = 0 \Longrightarrow -14k_1 = 0 \\ -3k_1 + k_2 & = 0 \Longrightarrow k_2 = 3k_1 \\ -k_1 + 2k_2 + k_3 & = 0 \Longrightarrow k_3 = -5k_1 \end{cases}$$

Hence, $\boxed{k_1 = k_2 = k_3}$

12. Consider the fourth O.D.E $y^{(4)} + y'' = 0$ and the functions $1, t, cos(t), sin(t)$. These are solutions since:

$$1^{(4)} + 1'' = 0 + 0 = 0$$

$$t^{(4)} + t'' = 0 + 0 = 0$$

$$cos(t)^{(4)} + cos(t)'' = cos(t) - cos(t) = 0$$

$$sin(t)^{(4)} + sin(t)'' = sin(t) - sin(t) = 0$$

The Wronskian is:

$$W(1, t, cos(t), sin(t)) = \begin{vmatrix} 1 & t & cos(t) & sin(t) \\ 0 & 1 & -sin(t) & cos(t) \\ 0 & 0 & -cos(t) & -sin(t) \\ 0 & 0 & sin(t) & -cos(t) \end{vmatrix} = \begin{vmatrix} 1 & -sin(t) & cos(t) \\ 0 & -cos(t) & -sin(t) \\ 0 & sin(t) & -cos(t) \end{vmatrix} = sin^2(t) + cos^2(t) = 1$$

15. Consider the third O.D.E $xy''' - y'' = 0$ and the functions $1, x, x^3$. These are solutions since:

$$x(1)''' - (1)'' = 0 - 0 = 0$$

$$x(x)''' - (x)'' = 0 - 0 = 0$$

$$x(x^3)''' - (x^3)'' = 6x - 6x = 0$$

The Wronskian is:

$$W(1, x, x^3) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = \begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} = 6x$$