M343 Homework 2

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Section 2.3

1. Let $Q(t)$ = the amount of dye in grams in the tank at time t. (Time in minutes). We want to find:

$$\frac{dQ}{dt} = \text{rate of dye into the tank} - \text{ rate of dye out of the tank}$$

Since we want to clean the tank, the rate of dye into the tank is zero. The rate of water in and out of the tank is the same: 2 L/min. The tank initially contains 200 L of a dye solution with a concentration of 1 g/L. The concentration of dye in the tank is $q(t) = Q(t)/V(t)$, but since the water flows in and out of the tank at the same rate we have that $V(t) = 200$ and hence, $q(t) = Q(t)/200$. A model for this situation is the following:

$$\begin{cases} \frac{dQ}{dt} = 0 \text{ g/L} \cdot 2 \text{ L/min} - \frac{Q(t)}{200} \text{ g/L} \cdot 2 \text{ L/min} = -\frac{Q(t)}{100} \text{ g/min} \\ Q(0) = 200 \text{ L} \cdot 1 \text{ g/L} = 200g \end{cases}$$

This model is an I.V.P consisting of a 1st order, linear differential equation solvable by separating variables. To solve it we proceed as follow:

$$\begin{aligned} \frac{dQ}{dt} &= -\frac{Q(t)}{100} && \text{Tank Model} \\ \frac{dQ}{Q} &= -\frac{dt}{100} && \text{Separating the equation} \\ \int \frac{dQ}{Q} &= \int -\frac{dt}{100} && \text{Integrating both sides} \\ ln(|Q|) &= -\frac{t}{100} + C && \text{Simple integration} \\ Q &= Ce^{-t/100} && \text{Exponentiating each side} \end{aligned}$$

Using our initial condition: $Q(0) = 200 = Ce^0 \Longrightarrow C = 200$. So our model for this particular situation is

$$Q(t) = 200e^{-t/100}$$

We are interested in finding the time t_0 such that the concentration of dye in the tank reaches 1% of its original value, i.e., $Q(t_0) = 1\% \cdot 200 = 2$. Using our model:

$$Q(t_0) = 2 = 200e^{-t_0/100} \iff \frac{1}{100} = e^{-t_0/100} \iff ln(\frac{1}{100}) = -t_0/100 \iff t_0 = 100ln(100) \text{ min}$$

2. Let $Q(t)$ = amount of salt in grams in the tank at time t. (Time in minutes). We want to find:

$$\frac{dQ}{dt} = \text{rate of salt into the tank} - \text{ rate of salt out of the tank}$$

The initial concentration of salt if γ g/L. The in and out rate of salt mixture to the tank is the same 2 L/min. Initially, the tank contains 120 L of pure water. Since the in and out rate is the same, the concentration of salt is $q(t) = Q(t)/V(t)$ where $V(t) = 120$ L. A model for this situation is the following:

$$\begin{cases} \frac{dQ}{dt} = \gamma \text{ g/L} \cdot 2 \text{ L/min} - \frac{Q(t)}{120} \text{ g/L} \cdot 2 \text{ L/min} = 2\gamma - \frac{Q(t)}{60} \\ Q(0) = 120 \text{ L} \cdot 0 \text{ g/L} = 0g \end{cases}$$

This model is an I.V.P consisting of a 1st order, linear differential equation solvable by integrating factor. To solve it we proceed as follow:

(i) Rewrite the equation as $Q' + \frac{1}{60}Q = 2\gamma$, (standard linear form).

(ii) Integrating factor: since $p(t) = \frac{1}{60}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int 1/60 dt} = e^{t/60}$

(iii) Multiply both sides of the equation by the integrating factor: $e^{t/60}[Q' + \frac{1}{60}Q = 2\gamma]$

(vi) Using product rule and implicit differentiation: $\frac{d}{dt}[e^{t/60}Q] = 2\gamma e^{t/60}$

(v) Integrate both sides: $\int \frac{d}{dt}[e^{t/60}Q]dt = \int 2\gamma e^{t/60}dt \Longrightarrow e^{t/60}Q = 120\gamma e^{t/60} + C$

The general solution is $Q(t) = 120\gamma + Ce^{-t/60}$. Solving for C using the initial condition: $Q(0) = 0 = 120\gamma + C \Longrightarrow C = -120\gamma$. The final model for the tank is:

$$Q(t) = 120\gamma - \frac{120\gamma}{e^{t/60}}$$

As time goes to infinity, the amount of salt in the tank goes to:

$$\lim_{t\to\infty} Q(t) = 120\gamma + 0 = 120\gamma$$

Since the term $\frac{120\gamma}{e^{t/60}}$, goes to zero as the exponent blows up quickly.

4. Let $Q(t)$ = amount of salt in lbs at time t. (Time in minutes). We want to find:

$$\frac{dQ}{dt} = \text{rate of salt into the tank} - \text{ rate of salt out of the tank}$$

The rate of mixture into the tank is 3 gal/min while the rate of mixture out of the tank is 2 gal/min. The mixture into the tank contains 1 lb of salt per gallon. In this case, the concentration of salt $q(t)$ varies according to $q(t) = Q(t)/V(t)$, where the volume at time t is given by the differential equation: $\frac{dV}{dt} = 3 - 2 = 1$, and hence, $V(t) = t + C$ (solving for initial condition $V(0) = 200 = C$). So the volume is given by $V(t) = t + 200$. A model for the change in the amount of salt in the tank is the following:

$$\begin{cases} \frac{dQ}{dt} = 3 \text{ gal/min} \cdot 1 \text{ lbs/gal} - \frac{Q(t)}{t+200} \text{ lbs/gal} \cdot 2 \text{ gal/min} = 3 - \frac{2}{t+200}Q \\ \\ Q(0) = 100 \text{ lbs} \end{cases}$$

This model is an I.V.P consisting of a 1st order, linear differential equation solvable by integrating factor. To solve it we proceed as follow:

(i) Rewrite the equation as $Q' + \frac{2}{t+200}Q = 3$, (standard linear form).

(ii) Integrating factor: since $p(t) = \frac{2}{t+200}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int 2/(t+200)dt} = (t+200)^2$

(iii) Multiply both sides of the equation by the integrating factor: $(t+200)^2[Q' + \frac{2}{t+200}Q = 3]$

(vi) Using product rule and implicit differentiation: $\frac{d}{dt}[(t+200)^2 Q] = 3(t+200)^2$

(v) Integrate both sides: $\int \frac{d}{dt}[(t+200)^2 Q]dt = \int 3(t+200)^2 dt \Longrightarrow (t+200)^2 Q = (t+200)^3 + C$

The general solution is $Q(t) = (t + 200) + C(t + 200)^{-2}$. Solving for C using the initial condition:

$$Q(0) = 100 = 200 + C(200^{-2}) \Longrightarrow C = -4 \times 10^6$$

The final model for the tank is:

$$Q(t) = (t + 200) - 4 \times 10^6 (t + 200)^{-2}$$

The solution begins to overflow when the tank is full. This happens exactly at t_0 when $V(t_0) = 500 = t_0 + 200 \Longrightarrow t_0 = 300$ min. So the solution is valid in the interval $0 \leq t < 300$.

Also, the concentration of salt on the point of overflowing is given by

$$q(300) = Q(300)/V(300) = [(300+200)-4\times10^6(300+200)^{-2}]/[300+200] = (500-16)/(500) = \frac{484}{500} = \frac{121}{125} \text{ lb/gal}$$

Finally, the theoretical limiting concentration if the tank had infinite capacity is given by:

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \frac{(t + 200) - 4 \times 10^6 (t + 200)^{-2}}{t + 200} = \lim_{t\to\infty} 1 - \frac{4 \times 10^6}{(t + 200)^3} = 1 \text{ lb/gal}$$

Since, $\lim_{t\to\infty} \frac{4 \times 10^6}{(t + 200)^3} = 0$, (the polynomial function in the denominator goes to infinity as t goes to infinity). So, in theory the concentration of salt will match exactly with the concentration of salt entering the tank as times passes.

16. Let T = temperature of cup of coffee and T_e = exterior temperature. Then, by Newton's law of cooling:

$$\begin{cases} \dfrac{dT}{dt} = k(T - T_e) & \text{for some constant } k \\ T(0) = 200, T(1) = 190, T_e = 70 \text{ lbs} \end{cases}$$

We can rewrite this equation as $T' - kT = -k70$. This is a first order, linear equation solvable by int. factor:

(i) The equation is already in the desired form: $T' - kT = -k70$, (standard linear form).
(ii) Integrating factor: since $p(t) = -k$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int -kdt} = e^{-kt}$
(iii) Multiply both sides of the equation by the integrating factor: $e^{-kt}[T' - kT = -k70]$
(vi) Using product rule and implicit differentiation: $\frac{d}{dt}[e^{-kt}T] = -k70e^{-kt}$
(v) Integrate both sides: $\int \frac{d}{dt}[e^{-kt}T]dt = \int -k70e^{-kt}dt \Longrightarrow e^{-kt}T = 70e^{-kt} + C$

The general solution is $T(t) = 70+Ce^{kt}$. This equation has two unknowns: C and k. To solve for C we use the initial condition: $T(0) = 200 = 70+Ce^0 \Longrightarrow C = 130$. We update the general solution to $T(t) = 70+130e^{kt}$. Finally, solve for k using $T(1) = 190 = 70 + 130e^t \Longrightarrow \frac{120}{130} = e^k \Longrightarrow k = ln\left(\frac{12}{13}\right)$. Hence, the equation modeling the change in temperature for the cup of coffee is:

$$T(t) = 70 + 130\left(\frac{12}{13}\right)^t$$

The coffee reaches the temperature of 150 exactly at t_0, i.e.:

$$T(t_0) = 150 = 70 + 130\left(\frac{12}{13}\right)^{t_0} \Longleftrightarrow \frac{80}{130} = \left(\frac{12}{13}\right)^{t_0} \Longleftrightarrow t_0 = \frac{ln\left(\frac{8}{13}\right)}{ln\left(\frac{12}{13}\right)}$$

Section 2.4

1. $(t-3)y' + ln(t)y = 2t, \qquad y(1) = 2$. In order to determine an interval in which the solution of the given IVP is certain to exists, let us apply Theorem 2.4.1.

   First, write the O.D.E. in canonical form: $y' + \dfrac{ln(t)}{(t-3)}y = \dfrac{2t}{(t-3)}$, where $p(t) = \dfrac{ln(t)}{(t-3)}$ and $g(t) = \dfrac{2t}{(t-3)}$

   The function $p(t)$ is defined if $t > 0$ since it depends on $ln(t)$. Also, both $p(t)$ and $g(t)$ are continuos only if $t - 3 \neq 0 \iff t \neq 3$. The IVP gives conditions on $t_0 = 1$. Therefore, the interval where this IVP is certain to have a unique solution is $t \in (0, 3)$.

3. $y' + tan(t)y = sin(t), \qquad y(\pi) = 0$

   This O.D.E is already in the canonical form. Hence, $p(t) = tan(t)$ and $g(t) = sin(t)$.

   The function $tan(t)$ is continuous everywhere except in $\frac{\pi}{2} + n\pi$, for $n \in \mathbb{N}$, while the function $sin(t)$ is continuous everywhere. The IVP gives conditions on $t_0 = \pi$. Therefore, the interval where this IVP is certain to have a unique solution is $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

7. $y' = \dfrac{t-y}{2t+5y}$. Let $f(t,y) = \dfrac{t-y}{2t+5y}$. Then,

$$\frac{\partial f}{\partial y}(t,y) = \frac{-1}{2t+5y} = \frac{5(t-y)}{(2t+5y)^2} = \frac{-(2t+5y) - 5t + 5y}{(2t+5y)^2} = \frac{-7t}{(2t+5y)^2}$$

   The functions $f(t,y)$ and $\dfrac{\partial f}{\partial y}(t,y)$ are continuous polynomials functions of t and y and hence, they are continuous everywhere except where they are not defined. In this case, these function are not defined if $2t + 5y = 0$, corresponding to a line in the ty-plane. The hypothesis of Theorem 2.4.2 are satisfied everywhere but in this line $2t + 5y = 0$. By Theorem 2.4.2 we can conclude that if $2t + 5y \neq 0$ then there exists a unique solution to this O.D.E.

10. $y' = (t^2 + y^2)^{\frac{3}{2}}$. Let $f(t,y) = (t^2 + y^2)^{\frac{3}{2}}$. Then,

$$\frac{\partial f}{\partial y}(t,y) = \frac{3}{2}(t^2 + y^2)^{\frac{1}{2}} 2y = 3y(t^2 + y^2)^{\frac{1}{2}}$$

   The functions $f(t,y)$ and $\dfrac{\partial f}{\partial y}(t,y)$ are continuous polynomials functions of t and y and hence, they are continuous everywhere except where they are not defined. In this case, there seems to be the constrain that $t^2 + y^2 \geq 0$, but since both quantities t^2 and y^2 are never negative we can be sure that this constrain is satisfied. The hypothesis of Theorem 2.4.2 are satisfied everywhere in the ty-plane. Hence, we can conclude there exists a unique solution to this O.D.E. everywhere in the ty-plane.

Bernoulli assigned problems

(1) $y' + \dfrac{4}{x}y = x^3y^2, \qquad y(2) = -1, \qquad x > 0$. This is a first order, non-linear, differential equation. It fits the Bernoulli case when $n = 2$. Note that, since $y' = f(x,y) = x^3y^2 - \frac{4}{x}y$ is defined everywhere except when $x = 0$, and so is $\frac{\partial f}{\partial y} = 2x^3y - \frac{4}{x}$, theorem 2.4.2 gives us the existence of a solution in some interval containing the point $(2, -1)$. To determine this, we need to solve the equation:

   (i) Divide both sides of the equation by y^2: $\dfrac{y'}{y^2} + \dfrac{4}{xy} = x^3$

   (ii) Make the change: $u = y^{1-n} = y^{-1} \Longrightarrow u' = \dfrac{-1}{y^2}y'$, to obtain the equation on u:

$$-u' + \frac{4}{x}u = x^3$$

   This is a first order, linear differential equation. We solve this by integrating factor:

(1) Write the equation in canonical form (divide by -1): $u' - \dfrac{4}{x}u = -x^3$

(2) $\mu(x) = e^{\int p(x)dx}$ where $p(x) = -\frac{4}{x}$, hence $\mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4ln(x)} = x^{-4}$

(3) Multiply both sides by $\mu(x)$: $\mu[u' - \dfrac{4}{x}u = -x^3] \iff u'x^{-4} - \dfrac{4}{x^5} = -x^{-1}$

(4) By the product rule: $\dfrac{d}{dx}[ux^{-4}] = -x^{-1}$

(4) Integrate both sides: $\int \dfrac{d}{dx}[ux^{-4}]dx = \int -x^{-1}dx \iff ux^{-4} = \ln(x) + C$

The general solution, in u is: $u(x) = (c - ln(x))x^4$

(iii) Change back to y using the relation $u = y^{-1}$:

$$u(x) = (c - ln(x))x^4 \Longrightarrow y(x) = \frac{1}{x^4(c - ln(x))}$$

(iv) Solve for the initial condition $y(2) = -1 = \dfrac{1}{2^4(C - ln(2))} \Longrightarrow C = ln(2) - \dfrac{1}{16}$

The particular solution for this I.V.P is

$$y(x) = \frac{1}{x^4(ln(2) - \frac{1}{16} - ln(x))}$$

The interval of validity of the solution: by hypothesis, $x > 0$. Now, the solution $y(x) = \dfrac{1}{x^4(c - ln(x))}$ is valid everywhere except in two cases: $x = 0$ or $C - ln(x) = 0 \iff ln(x) = C \iff x = e^C$. So the interval of validity for x is (e^C, ∞). In our case, $C = ln(2) - \dfrac{1}{16} \approx 0.6306$, so for our particular solution the interval of validity for x is $(0.6306, \infty)$, which contains $x_0 = 2$.

(2) $y' = 5y + e^{-2x}y^{-2}, \qquad y(0) = 2$. This is a first order, non-linear, differential equation. It fits the Bernoulli case when $n = -2$. Note that, since both $y' = f(x, y) = 5y + e^{-2x}y^{-2}$ and $\frac{\partial f}{\partial y}$ are discontinuous at $y = 0$, theorem 2.4.2 guarantees the existence of a solution in some interval containing $(0, 2)$. To determine this, we need to solve the equation:

(i) Rewrite and multiply both sides of the equation by y^2: $y'y^2 - 5y^3 = e^{-2x}$

(ii) Make the change: $u = y^{1-n} = y^3 \Longrightarrow u' = 3y^2y'$, to obtain the equation on u:

$$\frac{u'}{3} - 5u = e^{-2x}$$

This is a first order, linear differential equation. We solve this by integrating factor:

(1) Write the equation in canonical form (multiply by 3): $u' - 15u = 3e^{-2x}$

(2) $\mu(x) = e^{\int p(x)dx}$ where $p(x) = -15$, hence $\mu(x) = e^{\int -15dx} = e^{-15x}$

(3) Multiply both sides by $\mu(x)$: $\mu[u' - 15u = 3e^{-2x}] \iff e^{-15x}[u' - 15u] = 3e^{-17x}$

(4) By the product rule: $\dfrac{d}{dx}[u \cdot e^{-15x}] = 3e^{-17x}$

(4) Integrate both sides: $\int \dfrac{d}{dx}[u \cdot e^{-15x}]dx = \int 3e^{-17x}dx \iff u \cdot e^{-15x} = -\dfrac{3}{17}e^{-17x} + C$

The general solution, in u is: $u(x) = Ce^{15x} - \dfrac{3}{17}e^{-2x}$

(iii) Change back to y using the relation $u = y^3$:

$$u(x) = Ce^{15x} - \frac{3}{17}e^{-2x} \Longrightarrow y(x) = (Ce^{15x} - \frac{3}{17}e^{-2x})^{\frac{1}{3}}$$

(iv) Solve for the initial condition $y(0) = 2 = (Ce^0 - \dfrac{3}{17}e^0)^{\frac{1}{3}} = (C - \dfrac{3}{17})^{\frac{1}{3}} \Longrightarrow 2^3 = C - \dfrac{3}{17} \Longrightarrow C = \dfrac{139}{17}$

The particular solution for this I.V.P is

$$y(x) = (\frac{139}{17}e^{15x} - \frac{3}{17}e^{-2x})^{\frac{1}{3}}$$

For the interval of validity, not that the solution is defined everywhere since the cubic root is always continuos. Since our initial condition is $x_0 = 0$, the interval of validity for x is (∞, ∞) and for y $(0, \infty)$. So, the solution is valid in the upper-half of the $xy-$ plane, excluding the $x-$axis.

(3) $y' + \dfrac{y}{x} - \sqrt{y} = 0, \qquad y(1) = 0$. This is a first order, non-linear, differential equation. It fits the Bernoulli case when $n = \frac{1}{2}$. We solve it as follow:

(i) Rewrite and multiply both sides of the equation by $y^{-\frac{1}{2}}$: $\qquad y'y^{-\frac{1}{2}} + y^{\frac{1}{2}}x^{-1} = 1$

(ii) Make the change: $u = y^{1-n} = y^{\frac{1}{2}} \Longrightarrow 2u' = y'y^{-\frac{1}{2}}$, to obtain the equation on u:

$$2u' + \frac{u}{x} = 1$$

This is a first order, linear differential equation. We solve this by integrating factor:

(1) Write the equation in canonical form (divide by 2): $u' + \dfrac{1}{2x}u = \dfrac{1}{2}$

(2) $\mu(x) = e^{\int p(x)dx}$ where $p(x) = \frac{1}{2x}$, hence $\mu(x) = e^{\int \frac{1}{2x}dx} = e^{ln(x^{\frac{1}{2}})} = x^{\frac{1}{2}}$

(3) Multiply both sides by $\mu(x)$: $\mu[u' + \dfrac{1}{2x}u = \dfrac{1}{2}] \iff \mu[u' + \dfrac{1}{2x}u] = \dfrac{x^{\frac{1}{2}}}{2}$

(4) By the product rule: $\dfrac{d}{dx}[x^{\frac{1}{2}} \cdot u] = \dfrac{x^{\frac{1}{2}}}{2}$

(4) Integrate both sides: $\int \dfrac{d}{dx}[x^{\frac{1}{2}} \cdot u]dx = \int \dfrac{x^{\frac{1}{2}}}{2}dx \Longrightarrow x^{\frac{1}{2}} \cdot u = \frac{1}{3}x^{\frac{3}{2}} + C$

The general solution, in u is: $u(x) = \frac{1}{3}x + Cx^{-\frac{1}{2}}$

(iii) Change back to y using the relation $u = y^{\frac{1}{2}}$:

$$u(x) = \frac{1}{3}x + Cx^{-\frac{1}{2}} \Longrightarrow y(x) = (\frac{1}{3}x + Cx^{-\frac{1}{2}})^2$$

(iv) Solve for the initial condition $y(1) = 0 = (\frac{1}{3} + C)^2 \Longrightarrow C = -\dfrac{1}{3}$

The particular solution for this I.V.P is

$$y(x) = (\frac{1}{3}(x - \sqrt{x}))^2$$

The interval of validity of this solution for x is $(0, \infty)$ and for y is $[0, \infty)$ (first quadrant of the $xy-$plane); since $\sqrt{x}$ and $\sqrt{y}$ cannot be negative and this quadrant contain our initial condition $(x_0, y_0) = (1, 0)$.