M343 Homework 1

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Section 1.3

2. Second order, nonlinear.

4. First order, nonlinear.

6. Third order, linear.

13. Given the following second order, linear differential equation: $y'' + y = sec(t), \quad 0 < t < \pi/2$, let us check that the function $y(t) = cos(t)ln(cos(t)) + tsint(t)$ is a solution. For this, let us first compute y'' as follow:

$$\begin{aligned}
y'' &= [(cos(t)ln(cos(t)) + tsin(t))']' && \text{Definition of } y \\
&= [(cos(t)ln(cos(t)))' + (tsin(t))']' && \text{Linearity of derivative} \\
&= [-sin(t)ln(cos(t)) + cos(t)\tfrac{1}{cos(t)} - sin(t) + sin(t) + tcos(t)]' && \text{Product rule} \\
&= [(ln(cos(t)) + 1)(-sin(t)) + sin(t) + tcos(t)]' && \text{Grouping terms} \\
&= [(ln(cos(t)) + 1)(-sin(t))]' + sin(t)' + (tcos(t))' && \text{Linearity of derivative} \\
&= \tfrac{1}{cos(t)}sin^2(t) + (ln(cos(t)) + 1)(-cos(t)) + cos(t) + cos(t) + t(-sint(t)) && \text{Product rule} \\
&= \tfrac{sin^2(t)}{cos(t)} - cos(t)ln(cos(t)) - cos(t) + 2cos(t) - tsin(t) && \text{Rearranging terms} \\
&= \tfrac{sin^2(t)}{cos(t)} - cos(t)ln(cos(t)) + cos(t) - tsin(t) && \text{Rearranging terms}
\end{aligned}$$

Now, the relation becomes:

$$\begin{aligned}
y'' + y &= (\tfrac{sin^2(t)}{cos(t)} - cos(t)ln(cos(t)) + cos(t) - tsin(t)) + (cos(t)ln(cos(t)) + tsint(t)) \\
&= \tfrac{sin^2(t)}{cos(t)} + cos(t) && \text{Canceling terms} \\
&= \tfrac{sin^2(t)+cos^2(t)}{cos(t)} && \text{Trigonometric identity} \\
&= \tfrac{1}{cos(t)} && \text{Trigonometric identity} \\
&= sec(t). && \text{Showing that indeed } y \text{ is a solution to the differential equation.}
\end{aligned}$$

Section 2.2

5. $\dfrac{dy}{dx} = (cos^2(x))(cos^2(2y))$. This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned}
\frac{dy}{cos^2(2y)} &= cos^2(2x)dx && \text{Separating the equation} \\
\int \frac{dy}{cos^2(2y)} &= \int cos^2(2x)dx && \text{Integrating both sides} \\
x/2 + sin(2x)/4 &= sin(2y)/2cos(2y) + C && \text{Trigonometrical integration} \\
2x + sin(2x) &= 2tan(2y) + C && \text{Multiplying by 4 and trig. identities} \\
2x + sin(2x) - 2tan(2y) &= C && \text{General Solution.}
\end{aligned}$$

Where $cos(2y) \neq 0$, and $2y$ cannot be a multiple of $\pi/2$.

8. $\dfrac{dy}{dx} = \dfrac{x^2}{1+y^2}$. This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned}
(1+y^2)dy &= x^2dx && \text{Separating the equation}\\
\int(1+y^2)dy &= \int x^2dx && \text{Integrating both sides}\\
y+y^3/3 &= x^3/3+C && \text{Simple polynomial integration}\\
3y+y^3 &= x^3+C && \text{Multiplying by 3}\\
3y+y^3-x^3 &= C && \text{General Solution.}
\end{aligned}$$

12. Consider the differential equation: $\frac{dr}{d\theta} = \frac{r^2}{\theta}$ with initial condition $r(1) = 2$

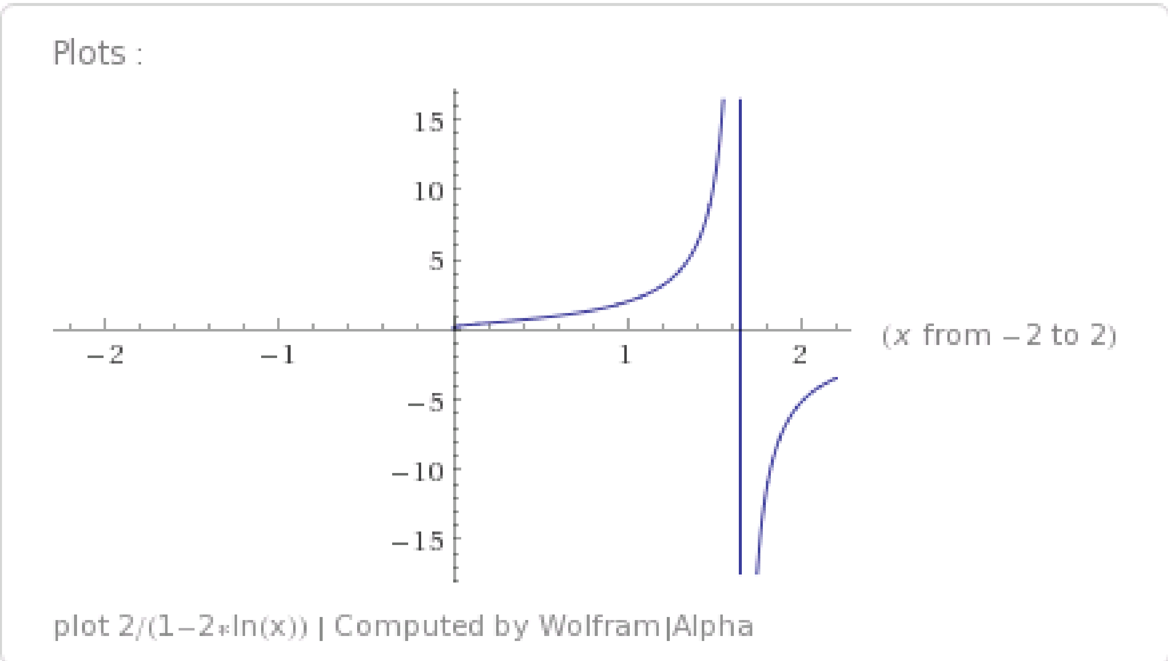
(a) This is a first order, nonlinear, separable equation. To solve it we proceed as follow:

$$\begin{aligned}
\frac{dr}{r^2} &= \frac{d\theta}{\theta} && \text{Separating the equation}\\
\int \frac{dr}{r^2} &= \int \frac{d\theta}{\theta} && \text{Integrating both sides}\\
\frac{-1}{r} &= ln(|\theta|)+C && \text{Solving for } r\\
r(\theta) &= \frac{-1}{ln(|\theta|)+C} && \text{General solution}
\end{aligned}$$

Now we solve for C given that $r(1) = 2 = \frac{-1}{ln(1)+C} = \frac{-1}{C} \Rightarrow C = \frac{-1}{2}$. Hence, the solution for the initial value problem is given explicitly by:

$$r(\theta) = \frac{2}{1-2ln(|\theta|)}$$

(b)



(c) The function $r(\theta)$ is a composition of continuos functions so it is continuos everywhere except where it is not defined. In this case, the function is not defined if $2ln(|\theta|) = 1$, which would make the denominator zero. Hence, $\theta \neq e^{\frac{1}{2}}$. The domain is therefore $\theta \in (0, \inf) \setminus e^{\frac{1}{2}}$.

15. Consider the differential equation: $y' = \dfrac{2x}{1+2y}$ with initial condition $y(x=2) = 0$.

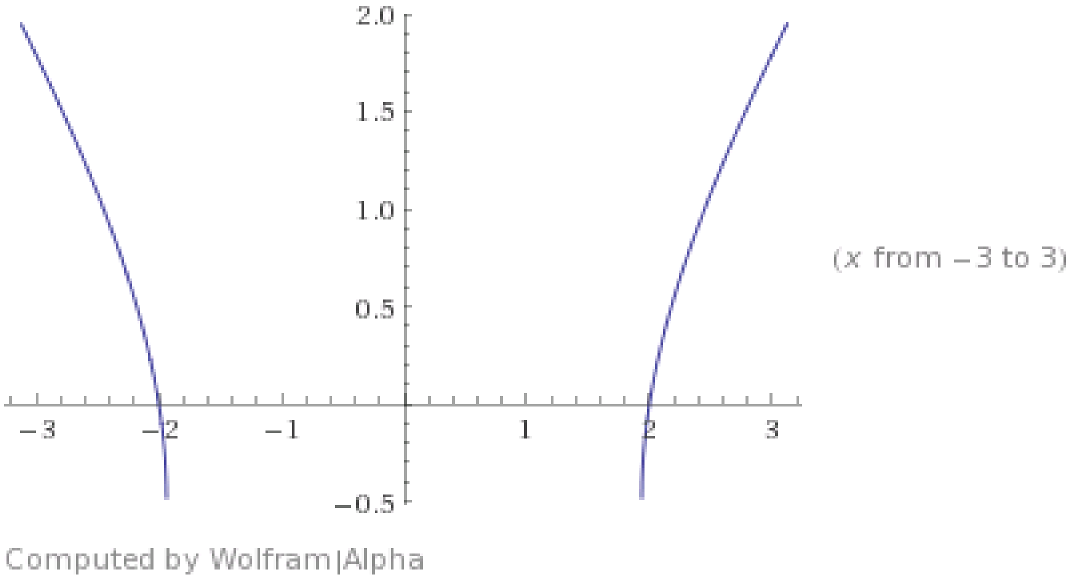
(a) This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned} (1+2y)dy &= 2xdx && \text{Separating the equation} \\ \int(1+2y)dy &= \int 2xdx && \text{Integrating both sides} \\ y+y^2 &= x^2+C && \text{Simple polynomial integration} \\ y+y^2-x^2 &= C && \text{General Solution.} \end{aligned}$$

Now we solve for C given that $y = 0$ when $x = 2$. Then, $0 + 0^2 - 2^2 = C \Rightarrow C = -4$. Finally,we solve y explicitly in terms of x to obtain the solution:

$$y+y^2-x^2 = -4 \iff (y+\frac{1}{2})^2 - \frac{1}{4} = x^2 - 4 \iff y + \frac{1}{2} = \pm\sqrt{x^2 - \frac{15}{4}} \iff y = \pm\sqrt{x^2-\frac{15}{4}} - \frac{1}{2}$$

(b)



(c) The solution $y = \pm\sqrt{x^2 - \frac{15}{4}} - \frac{1}{2}$ is defined if and only if $x^2 - \frac{15}{4} \geq 0 \iff x \geq \frac{\sqrt{15}}{2}$

19. Consider the differential equation: $sin(2x)dx + cos(3y)dy = 0$ with initial condition $y(x = \pi/2) = \pi/3$. This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned} -sin(2x)dx &= cos(3y)dy && \text{Separating the equation} \\ \int -sin(2x)dx &= \int cos(3y)dy && \text{Integrating both sides} \\ C - \tfrac{1}{2}cos(2x) &= -\tfrac{1}{3}sin(3y) && \text{Trigonometric integration} \\ \tfrac{1}{2}cos(2x) - \tfrac{1}{3}sin(3y) &= C && \text{General Solution.} \end{aligned}$$

Now we solve for C given that $y = \pi/3$ when $x = \pi/2$. Then, $\frac{1}{2}cos(\pi) - \frac{1}{3}sin(\pi) = C \Rightarrow C = \frac{1}{2}$. The solution is:

$$\frac{1}{2}cos(2x) - \frac{1}{3}sin(3y) = \frac{1}{2} \iff y(x) = \frac{arcsin(\frac{3}{2}(cos(2x)-1))}{3}$$

For the solution to be defined we have to have $cos^2(x) \leq 1/3 \iff |cos(x)| \leq 1/\sqrt{3}$, and hence,

$$arccos(1/\sqrt{3}) \leq x \leq \pi - arccos(1/\sqrt{3})$$

22. Consider the differential equation: $y' = \frac{3x^2}{3y^2-4}$ with initial condition $y(x=1) = 0$. This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned} (3y^2-4)dy &= 3x^2dx && \text{Separating the equation} \\ \int(3y^2-4)dy &= \int 3x^2dx && \text{Integrating both sides} \\ y^3-4y &= x^3+C && \text{Simple polynomial integration} \\ y^3-4y-x^3 &= C && \text{General Solution.} \end{aligned}$$

Now we solve for C given that $y=0$ when $x=1$. Then, $0^3-4(0)-1^3 = C \Rightarrow C = -1$. The solution is:

$$y^3-4y-x^3 = -1$$

To determine the interval in which the solution is valid we should first notice that from our original D.E., we have the constraint that $3y^2-4 \neq 0 \iff y \neq \pm\frac{2}{\sqrt{3}}$

23. Consider the differential equation: $y' = 2y^2 + xy^2$ with initial condition $y(x=0) = 1$. This is a first order, nonlinear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned} \frac{dy}{dx} &= y^2(2+x) && \text{Rewriting the equation} \\ \frac{dy}{y^2} &= (2+x)dx && \text{Separating the equation} \\ \int\frac{dy}{y^2} &= \int(2+x)dx && \text{Integrating both sides} \\ \tfrac{-1}{y} &= 2x+\frac{x^2}{2}+C && \text{Simple polynomial integration} \\ \tfrac{-2}{y} &= x^2+4x+C && \text{Multiplying both sides by 2} \\ y(x) &= \frac{-2}{x^2+4x+C} && \text{General Solution.} \end{aligned}$$

Now we solve for C given that $y(0) = 1 = \frac{-2}{C} \iff C = -2$. The solution is:

$$y(x) = \frac{-2}{x^2+4x-2}$$

To obtain the minimum value, it suffices to minimize $f(x) = x^2+4x-2$, since we take -2 and divided by this value. We know this is a parabola, and so it has a minimum value. Using the first and second derivative tests the minimum x_0 is such that: $f'(x_0) = 0 = 2x_0+4 \Rightarrow x_0 = -4/2 = -2$. Also, $f''(x_0) = 2 > 0$, hence x_0 is a minimum (which we already know from the fact that $f(x)$ is a parabola). Therefore, the minimum is attained at $x_0 = -2$.

24. Consider the differential equation: $y' = \frac{2-e^x}{3+2y}$ with initial condition $y(x=0) = 0$. This is a first order, linear, separable differential equation. To solve it we proceed as follow:

$$\begin{aligned} (3+2y)dy &= (2-e^x)dx && \text{Separating the equation} \\ \int(3+2y)dy &= \int(2-e^x)dx && \text{Integrating both sides} \\ 3y+y^2 &= 2x-e^x+C && \text{Simple polynomial integration} \end{aligned}$$

Now we solve for C given that $y=0$ when $x=0$. i.e., $0 = 0-e^0+C \Rightarrow C = 1$. Finally, we write the explicit solution:

$$3y + y^2 = 2x - e^x + 1 \iff \left(y + \frac{3}{2}\right)^2 = 2x - e^x + \frac{13}{4} \iff y = \pm\sqrt{2x - e^x + \frac{13}{4}} - \frac{3}{2}$$

This function is maximized exactly when $f(x) = \sqrt{2x - e^x + \frac{13}{4}}$ is maximize. Since the square root is an increasing function, we can maximize the simpler function $g(x) = f(x)^2 = 2x - e^x + \frac{13}{4}$. Using the first and second derivative tests, the minimum x_0 is such that : $f'(x_0) = 2 - e^{x_0} = 0 \Rightarrow e^{x_0} = 2 \iff x_0 = ln(2)$. Also, $f''(x_0) = -e^{x_0} < 0$ for any value of x_0. Hence, the maximum is attained at $x_0 = ln(2)$.

Section 2.1

3. (c) The equation $y' + y = te^{-t} + 1$ is a first order, linear equation. We solve this by integrating factor:

   (i) The equation is already in the desired form $y' + y = te^{-t} + 1$, with 1 as the coefficient of y'.

   (ii) Integrating factor: since $p(t) = 1$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int 1dt} = e^t$

   (iii) Multiply both sides of the equation by the integrating factor: $e^t[y' + y] = e^t[te^{-t} + 1]$

   (vi) Using product rule and implicit differentiation: $\frac{d}{dt}[e^t y] = te^{t-t} + e^t \Longrightarrow \frac{d}{dt}[e^t y] = t + e^t$

   (v) Integrate both sides: $\int \frac{d}{dt}[e^t y] = \int t + e^t dt \Longrightarrow e^t y = t^2/2 + e^t + C$

   The final solution is $y(t) = (t^2/2 + e^t + C)/e^t \iff y(t) = t^2/2e^t + 1 + C/e^t$.

   Also, since $\lim_{t\to\infty} y(t) = \infty^2/2e^\infty + 1 + C/e^\infty$ is an indeterminate form, we can use L'Hopital as follow:

$$\lim_{t\to\infty} y(t) = \frac{\frac{1}{2}t^2 + e^t + c}{e^t} = \lim_{t\to\infty} \frac{(\frac{1}{2}t^2 + e^t + c)'}{(e^t)'} = \lim_{t\to\infty} \frac{t + e^t}{e^t} \text{ still indeterminate apply L'Hopital again}$$

$$= \lim_{t\to\infty} \frac{t + e^t}{e^t} = \lim_{t\to\infty} \frac{(t + e^t)'}{(e^t)'} = \lim_{t\to\infty} \frac{1 + e^t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} + 1 = 0 + 1 = 1$$

   Since $1/e^t$ goes to zero as t goes to infinity. We could have also derived this limit by observing that e^t grows much faster than t^2.

7. (c) The equation $y' + 2ty = 2te^{-t^2}$ is a first order, linear equation. We solve this by integrating factor:

   (i) The equation is already in the desired form $y' + 2ty = 2te^{-t^2}$, with 1 as the coefficient of y'.

   (ii) Integrating factor: since $p(t) = 2t$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int 2tdt} = e^{t^2}$

   (iii) Multiply both sides of the equation by the integrating factor: $e^{t^2}[y' + 2ty] = e^{t^2}[2te^{-t^2}]$

   (vi) Using product rule and implicit differentiation: $\frac{d}{dt}[e^{t^2} y] = 2te^{t^2 - t^2} \Longrightarrow \frac{d}{dt}[e^{t^2} y] = 2t$

   (v) Integrate both sides: $\int \frac{d}{dt}[e^{t^2} y]dt = \int 2tdt \Longrightarrow e^{t^2} y = t^2 + C$

   The final solution is $y(t) = \frac{t^2 + C}{e^{t^2}}$.

   Also, since $\lim_{t\to\infty} y(t) = \frac{\infty^2 + C}{e^{\infty^2}}$ is an indeterminate form, we can use L'Hopital as follow:

$$\lim_{t\to\infty} y(t) = \frac{t^2 + C}{e^{t^2}} = \lim_{t\to\infty} \frac{(t^2 + C)'}{(e^{t^2})'} = \lim_{t\to\infty} \frac{2t}{2te^{t^2}} = \lim_{t\to\infty} \frac{1}{e^{t^2}} = 0$$

   We could have also derived this limit by observing that e^{t^2} grows much faster than $t^2 + C$ for C a constant.

8. (c) The equation $(1+t^2)y' + 4ty = (1+t^2)^{-2}$ is a first order, linear equation. We solve this by integrating factor:

   (i) Multiply by $(1 + t^2)^{-1}$ both sides of the equation to transform it to the desired form:

$$y' + \frac{4t}{1 + t^2} y = (1 + t^2)^{-3}$$

(ii) Integrating factor: since $p(t) = \dfrac{4t}{1+t^2}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{4t}{1+t^2}dt}$. We can solve the integral by making the substitution: $u = 1 + t^2 \Longrightarrow du = 2tdt \Longrightarrow dt = \frac{du}{2t}$, to obtain as the integrating factor: $e^{2ln(1+t^2)} = (1+t^2)^2$

(iii) Multiply both sides of the equation by the integrating factor: $(1+t^2)^2[y' + \dfrac{4t}{1+t^2}y] = (1+t^2)^2[(1+t^2)^{-3}]$

(vi) Using product rule and implicit differentiation: $\dfrac{d}{dt}[(1+t^2)^2 y] = (1+t^2)^{-1}$

(v) Integrate both sides: $\int \dfrac{d}{dt}[(1+t^2)^2 y]dt = \int (1+t^2)^{-1}dt \Longrightarrow (1+t^2)^2 y = arctan(t) + C$

The final solution is $y(t) = \dfrac{arctan(t) + C}{(1+t^2)^2}$.

Also, $\lim_{t\to\infty} y(t) = 0$ since $arctan(t)$ approaches $\pi/2$ as t goes to infinity, but $(1+t^2)^2$ approaches infinity as t approaches infinity. Again, we could have use L'Hopital to solve this limit.

15. The equation $ty' + 2y = t^2 - t + 1, y(1) = \frac{1}{2}, t > 0$ is a first order, linear equation. We solve this by integrating factor:

(i) Multiply by t^{-1} both sides of the equation to transform it to the desired form:

$$y' + \frac{2}{t}y = \frac{t^2 - t + 1}{t}$$

(ii) Integrating factor: since $p(t) = \dfrac{2}{t}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2ln(t)} = t^2$

(iii) Multiply both sides of the equation by the integrating factor: $t^2[y' + \dfrac{2}{t}y] = t^2[\dfrac{t^2 - t + 1}{t}] = t^3 - t^2 + t$

(vi) Using product rule and implicit differentiation: $\dfrac{d}{dt}[t^2 y] = t^3 - t^2 + t$

(v) Integrate both sides: $\int \dfrac{d}{dt}[t^2 y]dt = \int (t^3 - t^2 + t)dt \Longrightarrow t^2 y = \dfrac{t^4}{4} - \dfrac{t^3}{3} + \dfrac{t^2}{2} + C$

The general solution is $y(t) = \dfrac{\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C}{t^2} = \dfrac{t^2}{4} - \dfrac{t}{3} + \dfrac{1}{2} + \dfrac{C}{t^2}$.

Now we solve for the initial condition: $y(1) = \dfrac{1}{2} = \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{2} + C = \dfrac{3 - 4 + 6}{12} + C = \dfrac{5}{12} + C \Longrightarrow C = \dfrac{1}{12}$.

So our particular solution is:

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{\frac{1}{12}}{t^2} \iff y(t) = \frac{3t^4 - 4t^3 + 6t^2 + 1}{12t^2}$$

16. The equation $y' + \dfrac{2}{t}y = \dfrac{cost}{t^2}$, $y(\pi) = 0, t > 0$ is a first order, linear equation. We solve this by integrating factor:

(i) The equation is already in the desired form $y' + \dfrac{2}{t}y = \dfrac{cost}{t^2}$, with 1 as the coefficient of y'.

(ii) Integrating factor: since $p(t) = \dfrac{2}{t}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2ln(t)} = t^2$.

(iii) Multiply both sides of the equation by the integrating factor: $t^2[y' + \dfrac{2}{t}y] = t^2[\dfrac{cost}{t^2}]$

(vi) Using product rule and implicit differentiation: $\dfrac{d}{dt}[t^2 y] = cost$

(v) Integrate both sides: $\int \dfrac{d}{dt}[t^2 y] = \int costdt \Longrightarrow t^2 y = sin(t) + C$

The general solution is $y(t) = \dfrac{sin(t) + C}{t^2}$. Finally, we solve for C using the initial conditions:

$$y(\pi) = 0 = \frac{sin(\pi) + C}{\pi^2} = \frac{C}{\pi^2} \Longrightarrow C = 0$$

The particular solution is:

$$y(t) = \frac{sin(t)}{t^2}$$

27. Consider the initial value problem:

$$y' + \frac{1}{2}y = 2cos(t), \qquad y(0) = -1, t > 0$$

This equation is a first order, linear equation. We solve this by integrating factor:

(i) The equation is already in the desired form $y' + \frac{1}{2}y = 2cos(t)$, with 1 as the coefficient of y'.

(ii) Integrating factor: since $p(t) = \dfrac{1}{2}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{1}{2}dt} = e^{\frac{t}{2}}$.

(iii) Multiply both sides of the equation by the integrating factor: $e^{\frac{t}{2}}[y' + \frac{1}{2}y] = e^{\frac{t}{2}}2cos(t)$

(vi) Using product rule and implicit differentiation: $\dfrac{d}{dt}[e^{\frac{t}{2}}y] = 2e^{\frac{t}{2}}cos(t)$

(v) Integrate both sides: $\int \dfrac{d}{dt}[e^{\frac{t}{2}}ydt] = \int 2e^{\frac{t}{2}}cos(t)dt$.
Now, using integration by parts,we can compute the right hand side integral and thus obtain the general solution $y(t) = \frac{4}{5}(2sin(t) + cos(t)) + \frac{C}{e^{\frac{t}{2}}}$

Finally, use the initial condition to find a particular solution:

$$y(0) = -1 = \frac{4}{5} + \frac{C}{e^0} = \frac{4}{5} + C \iff C = -1 - \frac{4}{5} = -\frac{9}{5}$$

The solution is:

$$y(t) = \frac{4}{5}(2sin(t) + cos(t)) - \frac{9}{5e^{\frac{t}{2}}}$$

28. Consider the initial value problem:

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \qquad y(0) = y_0$$

Let t_0 be a value for which the solution $y(t)$ touches, but does not cross, the t-axis. Then we know from previous calculus that $y(t_0) = 0, y'(t_0) = 0$ and $y''(t_0) \neq 0$ (the second relation holds since at t_0 we have and inflection point and the last relation holds since the graph is either concave upward or downward). Using this information in our original differential equation we can solve for t_0 as follow:

$$0 + \frac{2}{3}0 = 1 - \frac{1}{2}t_0 \iff t_0 = 2$$

Hence, we have the additional information $y(2) = y'(2) = 0$. Now we can solve the D.E.

The equation $y' + \frac{2}{3}y = 1 - \frac{1}{2}t$ is a first order, linear equation. We solve this by integrating factor:

(i) The equation is already in the desired form $y' + \frac{2}{3}y = 1 - \frac{1}{2}t$, with 1 as the coefficient of y'.

(ii) Integrating factor: since $p(t) = \dfrac{2}{3}$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{3}dt} = e^{\frac{2}{3}t}$.

(iii) Multiply both sides of the equation by the integrating factor: $e^{\frac{2}{3}t}[y' + \frac{2}{3}y] = e^{\frac{2}{3}t}[1 - \frac{1}{2}t]$

(vi) Using product rule and implicit differentiation: $\dfrac{d}{dt}[e^{\frac{2}{3}t}y] = e^{\frac{2}{3}t}[1 - \frac{1}{2}t]$

(v) Integrate both sides: $\int \frac{d}{dt}[e^{\frac{2}{3}t}y]dt = \int e^{\frac{2}{3}t}[1 - \frac{1}{2}t]dt$.
Now, using integration by parts, setting $u = t \implies du = dt; v = e^{\frac{2}{3}t} \implies dv = \frac{2}{3}e^{\frac{2}{3}t}$, we can compute the right hand side integral and thus obtain $\int e^{\frac{2}{3}t}[1 - \frac{1}{2}t]dt = \frac{-3}{8}e^{\frac{2}{3}t}(2t-7) + C$

The general solution is $y(t) = \frac{\frac{-3}{8}e^{\frac{2}{3}t}(2t-7) + C}{e^{\frac{2}{3}t}}$. Now we need to solve for y_0. From the data of the problem we kwon that

$$y(0) = y_0 = \frac{-3}{8}(-7) + C \iff C = y_0 - \frac{21}{8}$$

We can rewrite our D.E. as

$$y(t) = \frac{\frac{-3}{8}e^{\frac{2}{3}t}(2t-7) + (y_0 - \frac{21}{8})}{e^{\frac{2}{3}t}}$$

Finally, using the fact that $y(2) = 0$ we can solve for y_0:

$$y(2) = 0 = \frac{\frac{9}{8}e^{\frac{4}{3}} - \frac{21}{8} + y_0}{e^{\frac{4}{3}}} \implies y_0 = \frac{21 - 9e^{\frac{4}{3}}}{8} \approx -1.642876$$

30. Consider the initial value problem:

$$y' - y = 1 + 3sin(t), \quad y(0) = y_0$$

We want to find y_0 such that the solution to the D.E. remains finite as $t \to \infty$. First we need to solve this equation. This is a first order, linear equation. We solve this by integrating factor:

(i) The equation is already in the desired form $y' - y = 1 + 3sin(t)$, with 1 as the coefficient of y'.
(ii) Integrating factor: since $p(t) = -1$ we get $\mu(t) = e^{\int p(t)dt} = e^{\int -1dt} = e^{-t}$.
(iii) Multiply both sides of the equation by the integrating factor: $e^{-t}[y' - y] = e^{-t}[1 + 3sin(t)]$
(vi) Using product rule and implicit differentiation: $\frac{d}{dt}[e^{-t}y] = e^{-t} + 3e^{-t}sin(t)$
(v) Integrate both sides: $\int \frac{d}{dt}[e^{-t}y]dt = \int (e^{-t} + 3e^{-t}sin(t))dt$.
Now, using integration by parts, setting $u = sin(t) \implies du = cos(t)dt; v = e^{-t} \implies dv = -e^{-t}$, we can compute the right hand side integral and thus obtain

$$\int (e^{-t} + 3e^{-t}sin(t))dt = -e^{-t} - \frac{3}{2}e^{-t}(sin(t) + cos(t)) + C$$

And so the general solution is:

$$y(t) = Ce^t - \frac{3}{2}(sin(t) + cos(t)) - 1$$

Plugging the value for $t = 0$ and solving for C:

$$y(0) = y_0 = C - 1 - \frac{3}{2} = C - \frac{5}{2} \iff C = y_0 + \frac{5}{2}$$

The particular solution is:

$$y(t) = (y_0 + \frac{5}{2})e^t - \frac{3}{2}(sin(t) + cos(t)) - 1$$

Since we have a linear factor of e^t, the only way for this function to remain positive is if we set $y_0 = -\frac{5}{2}$, so that we make null the term involving e^t, i.e., the function

$$y(t) = -\frac{3}{2}(sin(t) + cos(t)) - 1$$

will remain finite at t goes to infinity.