M312- Fall 2013- HW8 - Enrique Areyan ~~100~~ 99 (1)

(1) Exercise 7.4.4. The Torus T can be represented parametrically by $\Phi: D \to \mathbb{R}^3$: $\Phi(\varphi,\theta) = ((R+\cos\varphi)\cos\theta, (R+\cos\varphi)\sin\theta, \sin\varphi)$; whe $D=[0,2\pi]\times[0,2\pi]$, $R>1$.

Let us show that $A(T) = (2\pi)^2 R$ in two ways:

(i) By using formula (3):

$$A(T) = \iint_D \sqrt{\left[\frac{\partial(x,y)}{\partial(\varphi,\theta)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(\varphi,\theta)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(\varphi,\theta)}\right]^2}\, d\varphi\, d\theta; \quad \text{where:}$$

$$\Phi' = \begin{bmatrix}\partial\Phi/\partial\varphi \\ \partial\Phi/\partial\theta\end{bmatrix} = \begin{bmatrix} -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \\ -(R+\cos\varphi)\sin\theta & (R+\cos\varphi)\cos\theta & 0\end{bmatrix}.$$

$$\left[\frac{\partial(x,y)}{\partial(\varphi,\theta)}\right] = -\sin\varphi(R+\cos\varphi)\cos^2\theta - \sin\varphi(R+\cos\varphi)\sin^2\theta = [-\sin\varphi(R+\cos\varphi)](\cos^2\theta+\sin^2\theta) = -\sin\varphi(R+\cos\varphi)$$

$$\left[\frac{\partial(y,z)}{\partial(\varphi,\theta)}\right] = -(R+\cos\varphi)\cos\theta\cos\varphi \; ; \quad \left[\frac{\partial(x,z)}{\partial(\varphi,\theta)}\right] = (R+\cos\varphi)\sin\theta\cos\varphi. \quad \text{Replacing for } A(T):$$

$$A(T) = \int_0^{2\pi}\int_0^{2\pi} \sqrt{[-\sin\varphi(R+\cos\varphi)]^2 + [-(R+\cos\varphi)\cos\theta\cos\varphi]^2 + [(R+\cos\varphi)\sin\theta\cos\varphi]^2}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{\sin^2\varphi(R+\cos\varphi)^2 + (R+\cos\varphi)^2\cos^2\theta\cos^2\varphi + (R+\cos\varphi)^2\sin^2\theta\cos^2\varphi}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(R+\cos\varphi)^2[\sin^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\theta\cos^2\varphi]}\, d\varphi\, d\theta$$

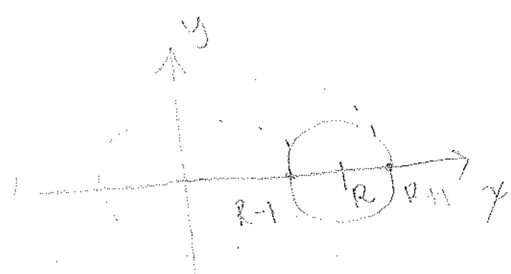
$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(R+\cos\varphi)^2[\sin^2\varphi + \cos^2\varphi(\cos^2\theta+\sin^2\theta)]}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(R+\cos\varphi)^2[\sin^2\varphi+\cos^2\varphi]}\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{2\pi}\sqrt{(R+\cos\varphi)^2}\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{2\pi} R+\cos\varphi\, d\varphi\, d\theta$$

$$= 2\pi\int_0^{2\pi} R+\cos\varphi\, d\varphi = 2\pi\left[R\varphi+\sin\varphi\right]_0^{2\pi} = 2\pi\left[(R\,2\pi + \underbrace{\sin(2\pi)}_{0}) - (R\cdot 0 + \underbrace{\sin(0)}_{0})\right] = \boxed{(2\pi)^2\cdot R}$$

(ii) By using formula (6):

$$A(T) = 2\pi\int_a^b \left(|x|\sqrt{1+[f'(x)]^2}\right)dx$$



We want to rotate the circle $(x-R)^2+y^2=1$ around the y-axis.

By symmetry, it suffices to rotate only the upper-half of the circle and double the result. Hence, our function to rotate is $f(x) = \sqrt{1-(x-R)^2}$ then;

$f'(x) = \frac{1}{2}\frac{1 \cdot -2(x-R)}{\sqrt{1-(x-R)^2}} = \frac{-(x-R)}{\sqrt{1-(x-R)^2}}$. Note that we will integrate ove positive values of x.

the area is given by:

$$A(T) = 2\int_{R-1}^{R+1} 2\pi x \sqrt{1+\left(\frac{-(x-R)}{\sqrt{1-(x-R)^2}}\right)^2}\,dx = 4\pi\int_{R-1}^{R+1} x\sqrt{1+\frac{(x-R)^2}{1-(x-R)^2}}\,dx = 4\pi\int_{R-1}^{R+1} x\sqrt{\frac{1-(x-R)^2+(x-R)^2}{1-(x-R)^2}}\,dx$$

$$= 4\pi\int_{R-1}^{R+1}\frac{x}{\sqrt{1-(x-R)^2}}\,dx$$; change: $u = x-R$ $du = dx$.

If $x = R-1$ then: $u = (R-1)-R = -1$

If $x = R+1$ then: $u = (R+1)-R = 1$.

$$= 4\pi\int_{-1}^{1}\frac{u+R}{\sqrt{1-u^2}}\,du = 4\pi\left[\int_{-1}^{1}\frac{u}{\sqrt{1-u^2}}\,du + \int_{-1}^{1}\frac{R}{\sqrt{1-u^2}}\,du\right] = 4\pi\left[-\sqrt{1-u^2} + R\arcsin(u)\right]_{-1}^{1}$$

$$= 4\pi\left[-\sqrt{1-1^2} + R\arcsin(1) + \sqrt{1-(-1)^2} - R\arcsin(-1)\right] = 4\pi\left[R(\arcsin(1) - \arcsin(-1))\right]$$

$$= 4\pi\left[R\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\right] = 4\pi^2 R = \boxed{(2\pi)^2 R}$$

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(2) Exercise 7.4.5. Let $\Phi(u,v) = (e^u\cos v, e^u\sin v, v)$; $\Phi : [0,1]\times[0,\pi] \to \mathbb{R}^3$.

(a) Find $T_u \times T_v$. First find T_u, then T_v and then $T_u \times T_v$.

$T_u = (e^u\cos v, e^u\sin v, 0)$; $T_v = (-e^u\sin v, e^u\cos v, 1)$.

$$T_u \times T_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^u\cos v & e^u\sin v & 0 \\ -e^u\sin v & e^u\cos v & 1 \end{vmatrix} = \hat{\imath}(e^u\sin v) - \hat{\jmath}(e^u\cos v) + \hat{k}(e^u) = \boxed{\langle e^u\sin v, -e^u\cos v, e^u\rangle}$$

e^{2u}

(b) Find the equation for the tangent plane to S when $(u,v) = (0,\frac{\pi}{2})$.

At this point, $\Phi(u,v) = \Phi(0,\frac{\pi}{2}) = (0, 1, \frac{\pi}{2})$. We need a vector perpendicular to this point. this is precisely $T_u\times T_v(0,\frac{\pi}{2}) = \langle 1, 0, 1\rangle$. the plane is given by:

$$T_u\times T_v(0,\pi/2)\cdot(\langle x,y,z\rangle - \langle 0,1,\tfrac{\pi}{2}\rangle) = 0 \iff$$

$$\langle 1,0,1\rangle\cdot\langle x, y-1, z-\tfrac{\pi}{2}\rangle = 0$$

$$\boxed{x + z - \frac{\pi}{2} = 0}$$

(c) Find the area of $\Phi(D)$.

$$\Phi' = \begin{bmatrix} e^u\cos v & e^u\sin v & 0 \\ -e^u\sin v & e^u\cos v & 1 \end{bmatrix}. \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = e^u; \quad \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = e^u\cos v; \quad \left|\frac{\partial(y,z)}{\partial(u,v)}\right| = e^u\sin v$$

the area A is given by:

$$A=\iint_D \sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2}\,du\,dv=\int_0^{\pi}\int_0^{1}\sqrt{(e^u)^2+(e^u\cos v)^2+(e^u\sin v)^2}\,du\,dv$$

$$=\int_0^{\pi}\int_0^{1}\sqrt{2e^{2u}}\,du\,dv=\pi\int_0^1\sqrt{2}\,e^u\,du=\sqrt{2}\pi\left[e^u\right]_0^1=\sqrt{2}\pi\left[e^1-e^0\right]=\boxed{\sqrt{2}\,\pi\,(e-1)}$$

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(3) Exercise 7.4.6. Find the area of the surface defined by $z=xy$ and $x^2+y^2\leq 2$.

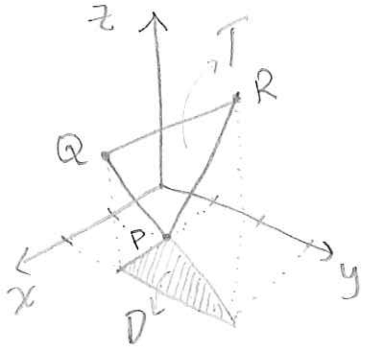
Solution: this is the area of the surface of a graph:

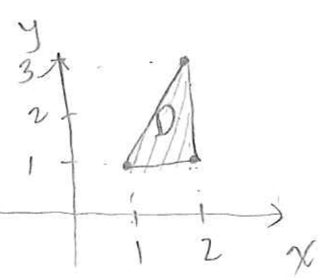
$$A(S)=\iint_D\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA=\iint_D\sqrt{y^2+x^2+1}\,dA.$$ changing to Polar coordinates:

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{r^2+1}\cdot r\,dr\,d\theta=2\pi\int_0^{\sqrt{2}}r\sqrt{r^2+1}\,dr;$$ change: $u=r^2+1$; $du=2r\,dr\Rightarrow r\,dr=\frac{du}{2}$

$$2\pi\int_\square^\square\sqrt{u}\,\frac{du}{2}=\pi\int_\square^\square\sqrt{u}\,du=\pi\left[\frac{2}{3}u^{3/2}\right]_\square^\square\rightsquigarrow\frac{2\pi}{3}\left[(r^2+1)^{3/2}\right]_0^{\sqrt{2}}=\frac{2\pi}{3}\left[3^{3/2}-1\right]=\boxed{\frac{2\pi}{3}\left[3\sqrt{3}-1\right]}$$

(4) Exercise 7.4.7. Consider the Triangle T in $\mathbb{R}^3$ with vertices: $(1,1,0)$, $(2,1,2)$ and $(2,3,3)$.





We want to find the surface area of T, which is the same as finding the surface area of the plane containing T over D.

First, let us find an equation for the plane containing T

$P=(1,1,0)$; $Q=(2,1,2)$; $R=(2,3,3)$

$PQ=(2,1,2)-(1,1,0)=(1,0,2)$.

$PR=(2,3,3)-(1,1,0)=(1,2,3)$.

the normal vector to the plane is:

$$\vec{n}=PQ\times PR=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&0&2\\1&2&3\end{vmatrix}=\hat{\imath}(-4)-\hat{\jmath}(1)+\hat{k}(2)=\langle-4,-1,2\rangle$$

the equation of the plane is:

$$\vec{n}\cdot(\langle x,y,z\rangle-\langle1,1,0\rangle)=0\iff$$

$$\langle-4,-1,2\rangle\cdot\langle x-1,y-1,z\rangle=0\iff-4x+4-y+1+2z=0$$

$$-4x-y+2z=-5\iff$$

$$\boxed{z=\frac{4x+y-5}{2}}$$

the surface area of T is given by:

$$A(T) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \iint_D \sqrt{4 + \frac{1}{4} + 1}\, dA = \iint_D \sqrt{\frac{21}{4}}\, dA = \frac{\sqrt{21}}{2} \iint_D dA,$$

where $\iint_D dA$ is just the area of the triangle D, i.e., $\frac{b \cdot h}{2} = \frac{1 \cdot 2}{2} = 1$.

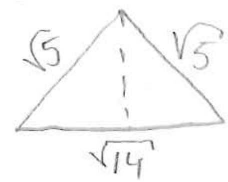
Hence, the area of T is $\boxed{A(T) = \frac{\sqrt{21}}{2}}$

Now, let us verify this answer by finding the lengths of the sides and using classical geometry: the lengths of T are:

$d(P,Q) = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ this is an isoceles triangle

$d(P,R) = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$

$d(Q,R) = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$



the base of the triangle is $\sqrt{14}$. the height can be obtained as follow:

$$(\sqrt{5})^2 = \left(\frac{\sqrt{14}}{2}\right)^2 + h^2 \Rightarrow 5 - \frac{14}{4} = h^2 \Rightarrow \frac{6}{4} = h^2 \Rightarrow h^2 = \frac{3}{2} \Rightarrow h = \frac{\sqrt{3}}{\sqrt{2}}$$

the area is: $\frac{h \cdot b}{2} = \frac{\sqrt{14} \cdot \frac{\sqrt{3}}{\sqrt{2}}}{2} = \frac{\sqrt{14 \cdot \frac{3}{2}}}{2} = \boxed{\frac{\sqrt{21}}{2}}$ the answer coincide.

(5) Exercise 7.4.19 Compute the area of the surface given by:

$x = r\cos\theta$; $y = 2r\cos\theta$; $z = \theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

Sketch.

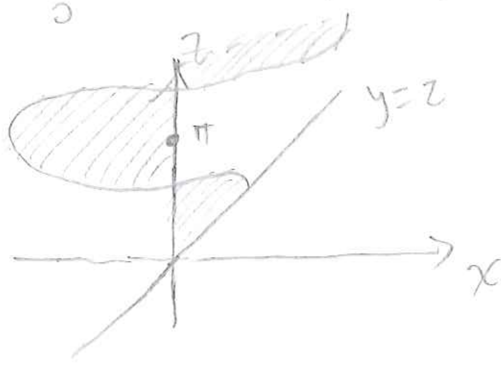
Solution: $\Phi' = \begin{pmatrix} \cos\theta & 2\cos\theta & 0 \\ -r\sin\theta & -2r\sin\theta & 1 \end{pmatrix}$. $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = -2r\sin\theta\cos\theta + 2r\sin\theta\cos\theta = 0.$

$\left|\frac{\partial(x,z)}{\partial(r,\theta)}\right| = \cos\theta$; $\left|\frac{\partial(y,z)}{\partial(r,\theta)}\right| = 2\cos\theta.$

$$A = \iint_D \sqrt{0^2 + \cos^2\theta + 4\cos^2\theta}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{5}\cos\theta\, dr\, d\theta = \sqrt{5}\int_0^{2\pi} \cos\theta\, d\theta.$$ Break into

4 pieces: $4\sqrt{5}\int_0^{\pi/2} \cos\theta = 4\sqrt{5}\left[\sin\theta\right]_0^{\pi/2} = \boxed{4\sqrt{5}}$

Sketch



(6) Exercise 7.4.25. Find the area of the graph of the function

$$f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$$

that lies over the domain $[0,1] \times [0,1]$.

Solution: Let us compute the surface area using:

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA, \text{ where:}$$

$$\frac{\partial f}{\partial x} = x^{1/2}, \quad \frac{\partial f}{\partial y} = y^{1/2} \Rightarrow A(S) = \iint_D \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1}\, dA$$

$$= \iint_D \sqrt{x+y+1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dx\,dy \quad ; \quad \text{change } u = x+y+1 \quad du = dx.$$

$$= \iint_{D^*} \sqrt{u}\, du\,dy = \int \frac{2}{3}\left[u^{3/2}\right] dy = \frac{2}{3}\int_0^1 \left[(x+y+1)^{3/2}\right]_{x=0}^{x=1} dy = \frac{2}{3}\int_0^1 (y+2)^{3/2} - (y+1)^{3/2}\, dy$$

$$= \frac{2}{3}\left[\frac{2}{5}(y+2)^{5/2} - \frac{2}{5}(y+1)^{5/2}\right]_0^1 = \frac{4}{15}\left[\left(3^{5/2} - 2^{5/2}\right) - \left(2^{5/2} - 1^{5/2}\right)\right]$$

$$= \frac{4}{15}\left[9\sqrt{3} - 4\sqrt{2} - 4\sqrt{2} - 1\right] = \boxed{\frac{4}{15}\left[9\sqrt{3} - 8\sqrt{2} - 1\right]}$$

(7) Exercise 7.5.4 Evaluate the integral

$$\iint_S (x+z)\, dS; \text{ where } S \text{ is the part of the cylinder } y^2 + y^2 = 4; \text{ with } x \in [0,5].$$

Solution: First we need to parametrize the cylinder. A possible parametrization is: $\Phi(u,\theta) = (u, 2\cos\theta, 2\sin\theta)$; $D: 0 \le u \le 5$; $0 \le \theta \le 2\pi$. By definition:

$$\iint_S (x+z)\, dS = \iint_D f(\Phi(u,\theta))\, \|T_u \times T_\theta\|\, du\, d\theta \,; \quad \text{where:}$$

$$\Phi' = \begin{pmatrix} \partial\Phi/\partial u \\ \partial\Phi/\partial\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{pmatrix}; \quad \left|\frac{\partial(x,y)}{\partial(u,\theta)}\right| = -2\sin\theta \,; \quad \left|\frac{\partial(x,z)}{\partial(u,\theta)}\right| = 2\cos\theta \,; \quad \left|\frac{\partial(y,z)}{\partial(u,\theta)}\right| = 0$$

$$\Rightarrow \|T_u \times T_\theta\| = \sqrt{(-2\sin\theta)^2 + (2\cos\theta)^2 + 0^2} = 2.$$

$$\iint_D f(\Phi(u,\theta))\|T_u \times T_\theta\|\, du\, d\theta = \int_0^{2\pi}\int_0^5 2(u + 2\sin\theta)\, du\, d\theta = 2\int_0^{2\pi} \left[\frac{u^2}{2} + 2u\sin\theta\right]_0^5 d\theta = 2\int_0^{2\pi} \frac{25}{2} + 10\sin\theta\, d\theta$$

$= \rightarrow$

$$2\int_0^{2\pi} \frac{25}{2}+10\sin\theta\, d\theta = 2\left[\frac{25}{2}\theta - 10\cos\theta\right]_0^{2\pi} = 2\left\{\left(25\pi-10\right)-\left(-10\right)\right\} = \boxed{50\pi}$$

(8) Exercise 7.5.5. Let S be the surface defined by $\Phi(u,v)=(u+v,u-v,uv)$.

(a) Show that the image of S is in the graph of the surface

$$4z = x^2-y^2.$$

Pf: To be in the graph, it has to satisfy the equation:

$$x^2-y^2=(u+v)^2-(u-v)^2 = u^2+2uv+v^2-(u^2-2uv+v^2) = 4uv = 4z$$

(b) Evaluate $\iint_S x\,dS$ for all points on the graph S over $x^2+y^2\le 1$.

Solution: By definition:

$$\iint_S x\,ds = \iint_D f(\Phi(u,v))\,\|T_u\times T_v\|\,du\,dv \; ; \text{ where } \Phi(u,v)=(u+v,u-v,uv). \quad D=\text{unit circle}$$

$$\Phi' = \begin{pmatrix} \frac{\partial\Phi}{\partial u} \\ \frac{\partial\Phi}{\partial v}\end{pmatrix} = \begin{pmatrix} 1 & 1 & v \\ 1 & -1 & u\end{pmatrix}; \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -2; \quad \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = u-v; \quad \left|\frac{\partial(y,z)}{\partial(u,v)}\right| = u+v.$$

$$\|T_u\times T_v\| = \sqrt{4+(u-v)^2+(u+v)^2} = \sqrt{4+u^2-2uv+v^2+u^2+2uv+v^2} = \sqrt{2u^2+2v^2+4}$$

$$\iint_D f(\Phi(u,v))\,\|T_u\times T_v\|\,du\,dv = \iint_D (u+v)\sqrt{2}\sqrt{u^2+v^2+2}\,du\,dv.$$ change to polar:

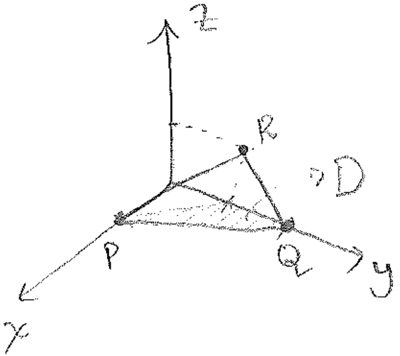
$$\sqrt{2}\int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta)\sqrt{r^2+2}\; r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\cos\theta\int_0^1 r^2\sqrt{r^2+2}\,dr + \sin\theta\int_0^1 r^2\sqrt{r^2+2}\,dr\right]d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[(\cos\theta+\sin\theta)\int_0^1 r^2\sqrt{r^2+2}\,dr\right]d\theta$$ Let $A=\int_0^1 r^2\sqrt{r^2+2}\,dr$. this is a number, (whatever it is). does not depend on θ.

$$= \sqrt{2}A\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta = \sqrt{2}A(\sin\theta-\cos\theta)\Big|_0^{2\pi} = \sqrt{2}A\left[(0-1)-(0-1)\right] = \boxed{0}$$

(9) Exercise 7.5.8. Evaluate $\iint_S xyz\,dS$ where S is the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,1,1)$

Solution: First we need to parametrize the triangle.



For that, let us find the equation of the plane containing P, Q and R.

$P=(1,0,0)$; $R=(0,1,1)$; $Q=(0,2,0)$.

$\vec{PQ}=(0,2,0)-(1,0,0)=(-1,2,0)$.

$\vec{PR}=(0,1,1)-(1,0,0)=(-1,1,1)$.

the normal vector to the plane containing the triangle is given by:

$$\vec{n}=\vec{PQ}\times\vec{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 0\\ -1 & 1 & 1\end{vmatrix}=\hat{i}(2)-\hat{j}(-1)+\hat{k}(-1+2)=\langle 2,1,1\rangle.$$

A unit normal vector to the plane is: $\hat{n}=\dfrac{\langle 2,1,1\rangle}{\sqrt{6}}$

the equation of the plane is:

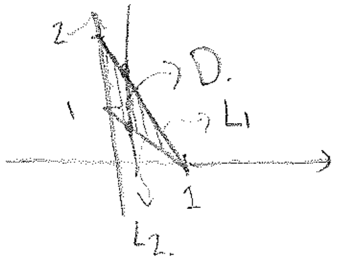
$$\vec{n}\cdot(\langle x,y,z\rangle-\langle 1,0,0\rangle)=0 \iff$$

$$\langle 2,1,1\rangle\cdot\langle x-1,y,z\rangle=0 \iff 2x-2+y+z=0$$

As a graph this plane can be written: $\boxed{z=2-2x-y}=g(x,y)$.

Next, we need to find a parametrization for the triangle. One such param. is

$\Phi(u,v)=\langle u,v,2-2u-v\rangle$ over the domain D given by:



L_1 is: $y=mx+b$: $2=m(0)+b \Rightarrow b=2$

$0=m(1)+b=m+2 \Rightarrow m=-2$

L_1: $\boxed{y=2-2x}$

L_2 is: $y=mx+b$: $1=m(0)+b\Rightarrow b=1$

$0=m(1)+b=m+1\Rightarrow m=-1$

$\boxed{y=1-x}$

$$\iint_S xyz\,dS=\iint_D f(x,y,g(x,y))\frac{dx\,dy}{\hat{n}\cdot\hat{k}}=\iint_D xy(2-2x-y)\frac{dx\,dy}{\frac{1}{\sqrt{6}}}=\sqrt{6}\int_0^1\int_{1-x}^{2-2x}xy(2-2x-y)\,dy\,dx$$

$$=\sqrt{6}\int_0^1\int_{1-x}^{2-2x}2xy-2x^2y-xy^2\,dy\,dx=\sqrt{6}\int_0^1\left[xy^2-(xy)^2-\frac{xy^3}{3}\right]_{1-x}^{2-2x}dx$$

$$=\sqrt{6}\int_0^1\left[y^2\left(x-x^2-\frac{xy}{3}\right)\right]_{1-x}^{2-2x}dx=\sqrt{6}\int_0^1(2-2x)^2\left(x-x^2-\frac{2x-2x^2}{3}\right)-(1-x)^2\left(x-x^2-\frac{x-x^2}{3}\right)dx$$

$$= \sqrt{6}\left[\int_0^1 (4-8x+4x^2)\left(\frac{x-x^2}{3}\right)dx - \int_0^1 (1-2x+x^2)\left(\frac{2x-2x^2}{3}\right)dx\right]$$

$$= \sqrt{6}\left[\frac{4}{3}\int_0^1 (1-2x+2x^2)(x-x^2)\,dx - \frac{2}{3}\int_0^1 (1-2x+x^2)(x-x^2)\,dx\right]$$

$$= \sqrt{6}\left[\frac{4}{3}\int_0^1 -2x^4+4x^3-3x^2+x\,dx - \frac{2}{3}\int_0^1 -x^4+3x^3-3x^2+x\,dx\right]$$

$$= \sqrt{6}\int_0^1 \frac{2}{3}x(1-x)^3\,dx = \sqrt{6}\int_0^1 \frac{2}{3}\cdot\frac{1}{4}(1-x)^4\,dx = \frac{\sqrt{6}}{5\cdot 6}\left[-(1-x)^5\right]_0^1 = \frac{\sqrt{6}}{30}\cdot-(0-1) = \boxed{\frac{\sqrt{6}}{30}}$$

(10) Exercise 7.5.13. Evaluate $\iint_S z\,dS$, where S is the surface $g(x,y) = z = x^2+y^2$, $x^2+y^2 \leq 1$.

Solution: $$\iint_S z\,dS = \iint_D f(x,y,g(x,y))\frac{dx\,dy}{\hat{n}\cdot\hat{k}}, \quad \text{where}$$

$$\vec{n} = \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle = \langle -2x, -2y, 1\rangle \Rightarrow \hat{n} = \frac{\langle -2x,-2y,1\rangle}{\sqrt{4x^2+4y^2+1}}$$

$$\iint_D x^2+y^2 \frac{dx\,dy}{\frac{1}{\sqrt{4x^2+4y^2+1}}} = \iint_D (x^2+y^2)\left(\sqrt{4x^2+4y^2+1}\right)dx\,dy$$ Changing to polar:

$$\int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\,r\,dr\,d\theta = 2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr;$$ which we can evaluate to:

$$= 2\pi\left[\frac{1}{120}(4r^2+1)^{3/2}(6r^2-1)\right]_0^1 = \frac{\pi}{60}\left[5^{5/2}+1\right] = \frac{\pi}{4}\left(\frac{5^{5/2}}{15}+\frac{1}{15}\right) = \frac{\pi}{4}\left(\frac{5^2\sqrt{5}}{15}+\frac{1}{15}\right)$$

$$= \boxed{\frac{\pi}{4}\left(\frac{5\sqrt{5}}{3}+\frac{1}{15}\right)}$$