EXAM 2, M312, Section 30353, 11/08/13

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Show your work. Simplify answers when possible. No books, notes, calculators are allowed. Use back sides as scratch paper (they will not be graded).

Do not write here

1 10 /10

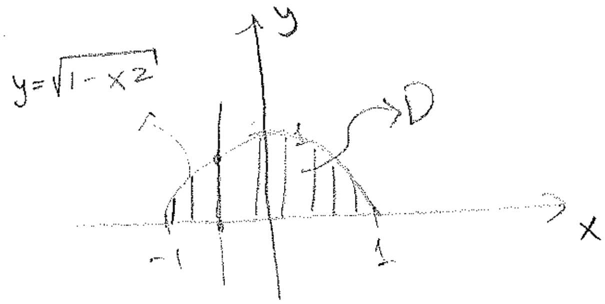
2 10 /10

3 10 /10

4 10 /10

5 10 /10 raw total 50 /50 standarized 100 /100 letter A+

1. (10 pts) Evaluate $\iint\limits_D xy^2\,dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \geq 0\}$.



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} xy^2\,dy\,dx = \int_{-1}^{1}\frac{x}{3}\left[y^3\right]_0^{\sqrt{1-x^2}}dx = \int_{-1}^{1}\frac{x}{3}\left(\sqrt{1-x^2}\right)^3dx$$

$$=\frac{1}{3}\int_{-1}^{1}x(1-x^2)^{3/2}\,dx$$

change variables to:

$u = 1-x^2 \Rightarrow du = -2x\,dx$

$\Rightarrow \frac{du}{-2} = x\,dx$

$$= \frac{1}{2}\int_{0}^{0}\frac{(u)^{3/2}}{-2}\,du = -\frac{1}{4}\left[\frac{2}{5}u^{5/2}\right]_0^0$$

changing back to x:

$$= -\frac{1}{10}\left[(1-x^2)^{5/2}\right]_{-1}^{1}$$

$$= -\frac{1}{10}\left[0-(0)\right] = \boxed{0}$$

2. (10 pts) Evaluate $\displaystyle\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, where $W = \{(x,y,z) \in \mathbb{R}^3 : 1 < x^2+y^2+z^2 < 4\}$.

Using spherical coordinates:

$x = \rho\cos\theta\sin\varphi$ the jacobian is
$y = \rho\sin\theta\sin\varphi$ $\rho^2\sin\varphi$.
$z = \rho\cos\varphi$

$$\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_0^{\pi}\int_0^{2\pi}\int_1^{2} \frac{\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}{(\rho^2)^{3/2}}$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^{2} \frac{\rho^2}{\rho^3}\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^{2} \frac{\sin\varphi}{\rho}\,d\rho\,d\theta\,d\varphi$$

$$= 2\pi\int_0^{\pi}\sin\varphi\,d\varphi\int_1^{2}\frac{1}{\rho}\,d\rho$$

$$= 2\pi\left[-\cos\varphi\right]_0^{\pi}\left[\ln(\rho)\right]_1^{2}$$

$$= 2\pi\left[-\cos\pi+\cos(0)\right]\left[\ln(2)-\underbrace{\ln(1)}_{0}\right]$$

$$= 2\pi\left[1+1\right]\ln(2)$$

$$= \boxed{4\ln(2)\pi}$$

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3. (10 pts) Let S be a surface in $\mathbb{R}^3$ given by the conditions $x^2+y^2+z^2=2$, $z \geq 1$. Find a parametrization of it corresponding to the representation of this surface as a graph of a function of x and y. Using it, find the area of this surface.

$x^2+y^2+z^2=2 \Rightarrow z=\pm\sqrt{2-x^2-y^2}$,

Since we are only interested in values such that $z \geq 1$.

We care only about the positive part:

$z=\sqrt{2-x^2-y^2}$, $z \geq 1$. where D is the projection onto xy-plane, i.e., $x^2+y^2+1^2=2 \Rightarrow x^2+y^2=1$

The parametrization is:

$\Phi(x,y)=(x,y,\sqrt{2-x^2-y^2})$, $0 \leq x^2+y^2 \leq 1$.

the area of the surface is given by:

$$\iint_S 1 \cdot dS = \iint_D \|\Phi_x \times \Phi_y\|\, dx\,dy, \quad \text{where:}$$

$$\Phi_x = \left(1, 0, \frac{-x}{\sqrt{2-x^2-y^2}}\right), \quad \Phi_y = \left(0, 1, \frac{-y}{\sqrt{2-x^2-y^2}}\right)$$

$$\Phi_x \times \Phi_y = \hat{i}\left(\frac{x}{\sqrt{2-x^2-y^2}}\right) - \hat{j}\left(\frac{-y}{\sqrt{2-x^2-y^2}}\right) + \hat{k}(1)$$

$$\Rightarrow \Phi_x \times \Phi_y = \left(\frac{x}{\sqrt{2-x^2-y^2}}, \frac{y}{\sqrt{2-x^2-y^2}}, 1\right)$$

$$\|\Phi_x \times \Phi_y\| = \sqrt{\frac{x^2+y^2}{2-x^2-y^2}+1} = \sqrt{\frac{x^2+y^2+2-x^2-y^2}{2-x^2-y^2}} = \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}. \text{ Hence,}$$

$$\text{Surface area} = \iint_D \frac{\sqrt{2}}{\sqrt{2-(x^2+y^2)}}\, dx\,dy = \sqrt{2}\iint_D \frac{1}{\sqrt{2-(x^2+y^2)}}\, dx\,dy, \text{ changing to Polar:}$$

$$\sqrt{2}\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{2-r^2}}\, r\,dr\,d\theta = 2\sqrt{2}\pi \int_0^1 \frac{r}{\sqrt{2-r^2}}\, dr$$

changing back for:

$u = 2-r^2 \Rightarrow du = -2r\,dr \Rightarrow \frac{du}{-2} = r\,dr$

$$\rightsquigarrow 2\sqrt{2}\pi \int_0^0 \frac{1}{\sqrt{u}}\frac{du}{-2} = -\sqrt{2}\pi \int_0^0 \frac{1}{\sqrt{u}}\, du = -\sqrt{2}\pi\, 2\left[\sqrt{u}\right]_0^0 \rightsquigarrow -2\sqrt{2}\pi\left[\sqrt{2-r^2}\right]_0^1$$

$$= -2\sqrt{2}\pi\left[\sqrt{2-1^2}-\sqrt{2}\right]$$

$$= -2\sqrt{2}\pi\left[1-\sqrt{2}\right] = \boxed{2\sqrt{2}\pi\left[\sqrt{2}-1\right]}$$

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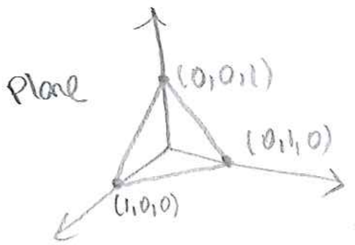
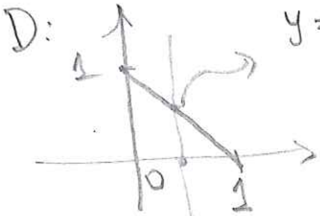
4. (10 pts) Find $\iint_S x\,dS$, where S is the part of the plane $x+y+z=1$ in the positive octant defined by $x \geq 0$, $y \geq 0$, $z \geq 0$.

First, we need to parametrize the plane : $z = 1-x-y$.





So, a parametrization for the plane is:

$\Phi(x,y) = (x, y, 1-x-y)$, where $0 \leq y \leq 1-x$, $0 \leq x \leq 1$

$\Phi_x = (1,0,-1)$
$\Phi_y = (0,1,-1)$ $\Rightarrow$ $\Phi_x \times \Phi_y = \hat{i}(1) - \hat{j}(-1) + \hat{k}(1) = \langle 1,1,1 \rangle$

$\|\Phi_x \times \Phi_y\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$. Therefore,

$$\iint_S x\,dS = \iint_D x\,\|\Phi_x \times \Phi_y\|\,dy\,dx = \sqrt{3}\int_0^1\int_0^{1-x} x\,dy\,dx = \sqrt{3}\int_0^1 x(1-x)\,dx$$

$$= \sqrt{3}\int_0^1 x - x^2\,dx = \sqrt{3}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \sqrt{3}\left[\frac{1}{2} - \frac{1}{3}\right] = \sqrt{3}\left[\frac{3-2}{6}\right] = \boxed{\frac{\sqrt{3}}{6}}$$

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5. (10 pts) Let $\mathbf{F} = (x, x^2, yz)$ represent the velocity field of a fluid (velocity measured in meters per second). Compute how many cubic meters of fluid per second are crossing the xy-plane through the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

the number of cubic meters of fluid per second is given by:

$$\iint_S \vec{F}\,d\vec{S} = \iint_D \vec{F} \cdot \Phi_x \times \Phi_y \, dx\,dy,$$

where Φ is a parametrization of the surface, in this case the xy-plane.

So, $\Phi(x,y) = (x, y, 0)$; $0 \leq x \leq 1, 0 \leq y \leq 1$.

$\Phi_x = (1,0,0)$
$\Phi_y = (0,1,0)$ $\Rightarrow$ $\Phi_x \times \Phi_y = (0,0,1)$. therefore, $\vec{F}(\Phi(x,y)) \cdot \Phi_x \times \Phi_y =$
$\langle x, x^2, y \cdot 0\rangle \cdot \langle 0,0,1\rangle$
$= x \cdot 0 + x^2 \cdot 0 + y \cdot 0 \cdot 1 = 0$

Hence:

$$\iint_D \vec{F} \cdot \Phi_x \times \Phi_y \, dx\,dy = \iint_D 0\,dx\,dy = 0 \iint_D dx\,dy = \boxed{0}$$

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