Math 53 Worksheet Solutions- Minmax and Lagrange

1. Find the local maximum and minimum values as well as the saddle point(s) of the function $f(x, y) = e^y(y^2 - x^2)$.

   Solution. First we calculate the partial derivatives and set them equal to zero.

   $$f_x(x, y) = -2xe^y = 0, \quad f_y(x, y) = e^y(y^2 - x^2) + 2ye^y = 0.$$

   For the first equation, $e^y \neq 0$, so we have $x = 0$. Plugging this in to the second equation, we have

   $$e^y[y^2 + 2y] = 0,$$

   and since again $e^y \neq 0$, we have $y = 0$ and $y = -2$. So our two critical points are $(0, 0)$ and $(0, -2)$. We now must check what kind of critical points these are. To do this we need the second and mixed partials,

   $$f_{xx}(x, y) = -2e^y, \quad f_{yy}(x, y) = e^y(y^2 - x^2 + 2y) + e^y(2y + 2), \quad f_{xy}(x, y) = -2xe^y.$$

   At the point $(0, 0)$, we have

   $$f_{xx}(0, 0) = -2, \quad f_{yy}(0, 0) = 2, \quad f_{xy}(0, 0) = 0,$$

   and so

   $$D = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = -4 < 0,$$

   and since $f_{xx}(0, 0) < 0$, we know that f has a local *maximum* at (0,0) by the second derivative test. At the point $(0, -2)$, we have

   $$f_{xx}(0, -2) = -2e^{-2}, \quad f_{yy}(0, -2) = -2e^{-2}, \quad f_{xy}(0, -2) = 0,$$

   and so

   $$D = f_{xx}(0, -2)f_{yy}(0, -2) - [f_{xy}(0, -2)]^2 = 4 > 0,$$

   so f has a *saddle point* at $(0, -2)$.

2. Find the absolute maximum and minimum values of $f(x, y) = 2x^3 + y^4$ on the set $D = \{(x, y)|x^2 + y^2 \leq 1\}$.

   Solution. Recall the method for find absolute maxima and minima:

   1. Find the critical points of f. No second derivative test is needed.
   2. Find the values of f at the function's critical points.
   3. Find the extreme values of f on the boundary of the region in question. (Don't forget endpoints!)
   4. The largest value from (2) and (3) is the maximum; the smallest is the minimum.

With these in mind, we're ready to attack the problem. So let's find the critical points by taking partials.

$$f_x(x,y) = 6x^2 = 0, \quad f_y(x,y) = 4y^3 = 0,$$

so $(x,y) = (0,0)$ is the only critical point. And note that $f(0,0) = 0$. Now we must look at the boundary, the circle $x^2 + y^2 = 1$. On this circle, $y^2 = 1 - x^2$, so define

$$g(x) = f(x,y) = 2x^3 + (1-x^2)^2,$$

and we will now find the extra of $g(x)$ as in traditional one-variable calculus. First calculate the derivative and set equal to zero.

$$g'(x) = 6x^2 - 4x(1-x^2) = 0 \Longrightarrow 2x[2x^2 + 3x - 2] = 0,$$

and so we have solutions $x = 0$ and

$$x = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{1}{2},\ -2,$$

but we can ignore the solution $x = -2$ as we seek to find the extrema of $g(x)$ on the interval $[-1,1]$. So we have

$$g(0) = 1, \quad g(\frac{1}{2}) = \frac{13}{16}.$$

But we still need to check the endpoints of g, since we're finding its extrema on a closed interval! Note that

$$g(-1) = -2, \quad g(1) = 2.$$

Looking back at all the values of $f(x,y)$, it turns out that the maximum occurs at $(1,0)$ and the minimum at $(-1,0)$. (Remember that $g(-1) = f(-1,0)$ and $g(1) = f(1,0)$.)

3. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution. The function we're finding extrema of is $f(x,y,z) = x^2y^2z^2$ and the constraint is $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. Then

$$\nabla f(x,y,z) = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle,$$

$$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle,$$

and so we must solve $\nabla f = \lambda \nabla g$. This equation is actually the three equations,

$$2xy^2z^2 = 2x\lambda, \quad 2yx^2z^2 = 2y\lambda, \quad 2zx^2y^2 = 2x\lambda,$$

along with the constraint equation. Observe that if any one of x, y, z is zero. Then $f(x,y,z) = 0$, according to the equation for $f(x,y,z)$. If none of the variables is equal to zero, then the equations become

$$\lambda = y^2z^2 = x^2z^2 = x^2y^2,$$

and because $x, y, z \neq 0$, this indicates that $x = y = z$. The constraint is then $3x^2 - 1 = 0$, or $x = \frac{1}{\sqrt{3}}$. In this case,

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^6 = \frac{1}{27}.$$

So the maximum occurs at $(x, y, z) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the minimum occurs at any point (x, y, z) where one of the variables is zero. The maximum value is $\frac{1}{27}$ and the minimum value is 0.

4. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

   Solution. The key here is that we needn't worry about the equation of the ellipse formed by the intersection; instead, we regard the two surfaces as constraints and seek to minimize the distance from the origin. More precisely, we'll minimize

$$f(x, y, z) = x^2 + y^2 + z^2,$$

   the distance squared, since this is minimized precisely when the distance is, and this function is a *lot* easier to work with. You should remember this trick. The constraints then are

$$g(x, y, z) = x + y + 2z - 2 = 0, \quad h(x, y, z) = x^2 + y^2 - z = 0.$$

   We must now solve

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

   So we calculate,

$$\nabla f = \langle 2x, 2y, 2z \rangle, \quad \nabla g = \langle 1, 1, 2 \rangle, \quad \nabla h = \langle 2x, 2y, -1 \rangle.$$

   And the equation we must solve becomes three equations, in addition to the two constraints.

$$2x = \lambda + 2x\mu, \quad 2y = \lambda + 2y\mu, \quad 2z = 2\lambda - \mu.$$

   Solving five equations for five unknowns can get tricky, to say the least. One idea is to observe that the first two equations above are quite similar. In fact, we can write

$$2x - 2x\mu = \lambda = 2y - 2y\mu \Longrightarrow (x - y)(2 - 2\mu) = 0.$$

   And so either $\mu = 1$ or $x = y$. First consider the case where $\mu = 1$. Then from the first equation above, $\lambda = 0$. Then $z = -\frac{1}{2}$. But then

$$x^2 + y^2 = -\frac{1}{2},$$

   and there are clearly no solutions in this case. So now assume $x = y$. At this point, we can dispense with solving for λ and μ entirely. (Remember, nothing about the problem insists we find their values!) So, going back to the original constraints, we have

$$x + z = 1, \quad 2x^2 = z.$$

These are much simpler! From the first equation, we get $z = 1 - x$, so the second then shows $2x^2 + x - 1 = 0$, and this factors as $(2x - 1)(x + 1) = 0$, so $x = \frac{1}{2}$ and $x = -1$ are solutions. So the two critical points are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(-1, -1, 2)$. Now we just calculate that

$$f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}, \quad f(-1, -1, 2) = 6,$$

and these are the minimum and maximum values, respectively. (Where minimum means the point closest to the origin and maximum is the point furthest.)

5. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Solution. It's not hard to see that if the side lengths are x, y, and z, then the length of the diagonal L and the volume V satisfy

$$L^2 = x^2 + y^2 + z^2, \quad V = xyz.$$

Now, we could use Lagrange multipliers to solve this problem, but let's do it first without such an advanced technique. To implement the constraint, we solve for z in the first equation and get

$$z = \sqrt{L^2 - x^2 - y^2},$$

so that we can write

$$V(x, y) = xy\sqrt{L^2 - x^2 - y^2}.$$

And now we just need to maximize V with the constraints $x, y > 0$ and $x^2 + y^2 \leq L^2$. So, first we find the critical points.

$$V_x(x, y) = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}}, \quad V_y(x, y) = x\sqrt{L^2 - x^2 - y^2} - \frac{xy^2}{\sqrt{L^2 - x^2 - y^2}}.$$

Setting the components equal to zero gives

$$\frac{1}{\sqrt{L^2 - x^2 - y^2}} \left[L^2 - 2x^2 - y^2\right] y = 0,$$

and

$$\frac{1}{\sqrt{L^2 - x^2 - y^2}} \left[L^2 - x^2 - 2y^2\right] x = 0.$$

Now $x, y \neq 0$, as noted above, so we must have

$$L^2 - 2x^2 - y^2 = 0, \quad L^2 - x^2 - 2y^2 = 0.$$

Write the first equation as $y^2 = L^2 - 2x^2$ and substitute into the second. We then have

$$L^2 - x^2 - 2(L^2 - 2x^2) = 0, \Longrightarrow 3x^2 = L^2,$$

and so $x = \frac{L}{\sqrt{3}}$. Substituting this into the first equation, it follows that $y = \frac{L}{\sqrt{3}}$. So the maximum volume is obtained when

$$x = y = z = \frac{L}{\sqrt{3}},$$

and the volume then is

$$V = \left(\frac{L}{\sqrt{3}}\right)^3 = \frac{L^3}{3\sqrt{3}}.$$

The problem is indeed easier with Lagrange. Here $f(x, y, z) = xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - L^2 = 0$. Then

$$\nabla f(x, y, z) = \langle yz, xz, xy\rangle, \quad \nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle,$$

and the equations are

$$yz = 2x\lambda, \quad xz = 2y\lambda, \quad xy = 2z\lambda.$$

Then

$$xyz = 2x^2\lambda = 2y^2\lambda = 2z^2\lambda,$$

which we obtain by multiplying each equation by x, y, and z respectively. Thus either $\lambda = 0$ or $x = y = z$, since $x, y, z > 0$. If $\lambda = 0$, then at least one of x, y, z is zero, which cannot happen. So we must have $x = y = z$ and, from the constraint equation, this means we must have

$$x = y = z = \frac{L}{\sqrt{3}}.$$

Then, again, we find that

$$V = \left(\frac{L}{\sqrt{3}}\right)^3 = \frac{L^3}{3\sqrt{3}}.$$