Math 53 Worksheet Solutions - Cross Products and Planes

1. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

   Solution.

$$\begin{aligned}(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \times (\mathbf{a} + \mathbf{b}) - \mathbf{b} \times (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} \\ &= 2(\mathbf{a} \times \mathbf{b})\end{aligned}$$

2. Use the scalar triple product to determine whether the points $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$, and $D(3, 6, -4)$ lie in the same plane.

   Solution. Set $\mathbf{a} = AB = \langle 2, -4, 4\rangle$, $\mathbf{b} = AC = \langle 4, -1, -2\rangle$, and $\mathbf{c} = AD = \langle 2, 3, -6\rangle$. We need to know if this vectors lie in the same plane. As such, we take the scalar triple product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 0.$$

   As the scalar triple product is zero, they are coplanar.

3. Find the distance from the point $(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$.

   Solution. Use the formula from the book. $a = 3$, $b = 2$, $c = 6$, $d = -5$, and $(x_1, y_1, z_1) = (1, -2, 4)$. So

$$D = \frac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{18}{7}.$$

4. Give a geometric description of each family of planes.

   (a) $x + y + z = c$

   (b) $x + y + cz = 1$

   (c) $y \cos\theta + z \sin\theta = 1$

   Solution.

   (a) Normal vector is $\mathbf{n} = \langle 1, 1, 1\rangle$. x, y, and z intercepts are all c. Draw a picture. Plane makes equilateral triangle in first octant ($c > 0$) or octant opposite the first ($c < 0$).

   (b) Normal vector is $\mathbf{n} = \langle 1, 1, c\rangle$. x an y intercepts are 1, but z intercept is $\frac{1}{c}$. As such, as c gets large, plane approaches the xy plane.

(c) Normal vector is $\mathbf{n} = \langle 0, \cos\theta, \sin\theta \rangle$. Note that an obvious point on the plane is $(x, \cos\theta, \sin\theta)$ for any x. It follows that the family consists of planes tangent to the cylinder with radius 1 and with the x axis as its major axis.

5. If a, b, and c are all not 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c \rangle$ is a normal vector to the plane.

Solution. Assume $a \neq 0$. The proof for the other cases is similar. Then we can simply move d into the x term, as in

$$a\left(x - \frac{d}{a}\right) + by + cz = 0,$$

and this now has the form

$$\langle a, b, c \rangle \cdot \langle x - \frac{d}{a}, y, z \rangle = 0,$$

which is the vector form of a plane.

6. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

Solution. The direction vectors of the lines are $\mathbf{v_1} = \langle 1, 6, 2 \rangle$ and $\mathbf{v_2} = \langle 2, 15, 6 \rangle$. So $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2}$ should be perpendicular to both lines. We calculate

$$\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} = \langle 6, -2, 3 \rangle.$$

Draw a picture of what this looks like. Now pick points on the lines, $P_1 = (1, 1, 0)$ and $P_2 = (1, 5, -2)$ will do. (Get this by setting $t = s = 0$.) Form the vector connecting these points on the lines

$$\mathbf{b} = \langle 0, 4, -2 \rangle.$$

Then, by a geometry argument or by scalar projection, it follows that

$$D = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{14}{\sqrt{6^2 + (-2)^2 + 3^2}} = \frac{14}{7} = 2.$$

7. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

Solution. We could use the next problem to do this easily, but there is an easier way. The normal vector to the plane is $\mathbf{n} = \langle 1, 2, -2 \rangle$. A vector in the same direction as the normal vector but with length 2 (check this) is

$$\mathbf{v} = \frac{\mathbf{n}}{|\mathbf{n}|}(2) = \langle \frac{2}{3}, \frac{4}{3}, -\frac{4}{3} \rangle.$$

A point on the plane is $P_0 = (1, 1, 1)$. The planes that are parallel and two units away will have identical normal vectors, but different points. Moving 2 units in the direction (or antidirection) of the normal vector from point P_0 will take us to new points

$$P_1 = P_0 + \mathbf{v} = (\frac{5}{3}, \frac{7}{3}, -\frac{1}{3}), \quad P_2 = P_0 - \mathbf{v} = (\frac{1}{3}, -\frac{1}{3}, \frac{7}{3}).$$

The new equations of the planes are then

$$(1)(x - \frac{5}{3}) + 2(y - \frac{7}{3}) + (-2)(z + \frac{1}{3}) = 0, \quad (1)(x - \frac{1}{3}) + (2)(y + \frac{1}{3}) + (-2)(z - \frac{7}{3}) = 0.$$

These can be rewritten

$$x + 2y - 2z = 7, \quad x + 2y - 2z = -5.$$

8. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Solution. Pick a point on the second plane, and call it $P_1 = (x_1, y_1, z_1)$. Then $ax_1 + by_1 + cz_1 + d_2 = 0$ and in fact

$$ax_1 + by_1 + cz_1 = -d_2.$$

The distance between the first plane and this point is given by the formula in the book, so

$$D = \frac{|ax_1 + by_1 + cz_1 + d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_2 + d_1|}{\sqrt{a^2 + b^2 + c^2}},$$

and this is the formula.

9. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}, \quad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the norm $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a lattice for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ form the reciprocal lattice.)

(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.

(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

Solution.

(a) For any vectors $\mathbf{a}$ and $\mathbf{b}$ the scalar triple products

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0, \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0,$$

so it follows that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ when $i \neq j$.

(b) Evaluate each one and use part (5) of Theorem 8.

(c) Perform the calculation using the properties of the cross product in Theorem 8.