Math 32 A: Practice Final Solutions

1. Use vectors to decide whether the triangle with vertices $P(1,-3,-2)$, $Q(2,0,-4)$ and $R(6,-2,-5)$ is right-angled.

   Solution: $PQ = \langle 1,3,-2\rangle, QR = \langle 4,-2,-1\rangle, PR = \langle 5,1,-3\rangle$. $PQ \cdot QR = 4-6+2 = 0$, so yes, the triangle is right-angled.

2. Reparameterize the curve $\mathbf{r}(t) = (t, \cos(2t), \sin(2t))$ with respect to arc length measured from the point $(0,1,0)$ in the direction of increasing t.

   Solution: $s(t) = \int_0^t |\mathbf{r}'(x)|dx = \int_0^t \sqrt{1+4\cos^2(2x)+4\sin^2(2x)}dx = t\sqrt{5}$. Then $t(s) = \frac{s}{\sqrt{5}}$, and so $\mathbf{r}(s) = (\frac{s}{\sqrt{5}}, \cos(2\frac{s}{\sqrt{5}}), \sin(2\frac{s}{\sqrt{5}}))$.

3. A projectile is shot from the origin with initial velocity $\mathbf{v}_0 = (u_0, v_0)$. Assuming that the projectile acceleration is only due to gravity (i.e. $\mathbf{a} = (0,-9.8)$), how large must v_0 be if the projectile is to reach a given height $h > 0$ at some time t?

   Solution: Since $\mathbf{r}''(t) = (0,-9.8)$, by integrating we see that $\mathbf{r}'(t) = (u_0, -gt+v_0)$, and integrating again we see that $\mathbf{r}(t) = (u_0 t, -\frac{g}{2}t^2 + v_0 t)$ (assuming the projectile starts at the origin). Since we are interested only in the height of the projectile, we only care about the second component. Thus we can set $h(t) = -\frac{g}{2}t^2 + v_0 t$, and the maximum height occurs at the time $t_{\max}$ that satisfies $h'(t) = -gt + v_0 = 0$, which is $t_{\max} = v_0/g$. Plugging back into the expression for the height, we see that in order for a height of h to be reached sometime, the maximum height that the projectile reaches must be greater than or equal to h; consequently we need $h(t_{\max}) = -\frac{g}{2}\left(\frac{v_0}{g}\right)^2 + \frac{v_0^2}{g} = \frac{v_0^2}{2g} \geq h \implies v_0 \geq \sqrt{2gh}$.

4. Find and sketch the domain of $f(x,y) = \ln(x+y)\sqrt{y-x}$.

   Solution: The domain $D(f)$ of $f(x,y) = g(x,y)h(x,y)$ is just $D(f) = D(g) \cap D(h)$. Since $g(x,y) = \log(x+y)$, $D(g) = \{(x,y) \in \mathbb{R}^2 | x+y > 0\}$, and since $h(x,y) = \sqrt{y-x}$, $D(h) = \{(x,y) \in \mathbb{R}^2 | y \geq x\}$, and so $D(f) = D(g) \cap D(h) = \{(x,y) \in \mathbb{R}^2 | -x < y, x \leq y\}$. Make sure you know what this looks like and can sketch it - it should look like a triangular wedge opening to the right.

5. Find the limit if it exists otherwise show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{\sin^2(x)}$$

   Solution: By restricting to the path $y = 0$ we get a limit of 1 (show this using L'Hopital's Rule!), and by restricting to the path $y = x^2$ we get a limit of $1/2$. Thus the limit does not exist.

6. a. Write the linearization $L(x, y)$ of the function $z = f(x, y) = x^3y^4$ at the point $(1, 1)$.

Solution: $L(x, y) = f(1, 1)+f_x(1, 1)(x-1)+f_y(1, 1)(y-1) = 1+3(x-1)+4(y-1) = 3x + 4y - 6$.

b. At what values t does the curve $\mathbf{r}(t) = (t^2, 3t + 1, t + 2)$ intersect the plane defined by the linearization in part a?

Solution: The plane defined by the linearization is just $3x + 4y - z = 6$. Plugging in the coordinates of the curve, we get $3t^2 + 11t - 4 = 0$. Using the quadratic formula, we get $t = -4, \frac{1}{3}$.

7. Define $h(u, v, w) = z(x(u, v, w), y(u, v, w))$ with $z(x, y) = x^2 + xy^3$, $x(u, v, w) = uv^2+w^3$, $y(u, v, w) = u+ve^w$. What are $\frac{\partial h}{\partial u}$,$\frac{\partial h}{\partial v}$ and $\frac{\partial h}{\partial w}$ when $u = 2$,$v = 1$ and $w = 0$?

Solution: First note that $x(2, 1, 0) = 2$ and $y(2, 1, 0) = 3$. Then we evaluate

$$\frac{\partial h}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + 3xy^2 = 31 + 54 = 85.$$

$$\frac{\partial h}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + 3xy^2(e^w) = 31(4) + 54 = 178.$$

$$\frac{\partial h}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + 3xy^2(ve^w) = 0 + 54 = 54.$$

8. Find the surface of revolution defined by rotating the curve $y^2 + \frac{z^2}{9} = 1$ about the z-axis.

Solution: Let S denote the surface of revolution. Then

$$S = \{(x, y, z) \in \mathbb{R}^3 | d\left((x, y, z), (0, 0, z)\right) = d\left((0, u, z), (0, 0, z)\right)\},$$

where u satisfies $u^2 + \frac{z^2}{9} = 1$. Solving this distance equation gives $x^2 + y^2 = u^2 = 1 - \frac{z^2}{9}$. The equation that S is determined by is thus $x^2 + y^2 + \frac{z^2}{9} = 1$, and this is an ellipsoid.

9. Two level surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ meet in a curve at the point (a,b,c). How would you find the tangent to the curve at the point (a, b, c)?

Solution: You could find the equation of the curve by solving $f(x, y, z) = g(x, y, z)$, then convert to parametric form. Another way would be to find the tangent planes to each surface at the point (a, b, c) and then find their line of intersection.

10. Use Lagrange multipliers to find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

    Solution: We have $\nabla f(x, y) = \lambda \nabla g(x, y)$, where $g(x, y) = x^2 + y^2$. Solving this, we get $\langle 2x, 4y\rangle = \lambda\langle 2x, 2y\rangle$. Initially it may seem that $\lambda = 0$ is a solution, but if it was we would get $\nabla f = 0 \implies x = y = 0$, which violates the constraint $g(x, y) = x^2 + y^2 = 1$. Thus $\lambda \neq 0$. Then we have the two equations $x = \lambda x$ and $y = \frac{\lambda y}{2}$. The first equation gives us that either $x = 0$ or $\lambda = 1$. If $x = 0$ then we have two possible solutions $(0, 1)$ and $(0, -1)$. If $\lambda = 1$ then $y = 0$ and we get two more possible solutions $(1, 0)$ and $(-1, 0)$. We check the values of the function at these four points, and see that the maximum of 2 is reached at $(0, 1)$ and $(0, -1)$, while the minimum of 1 is reached at $(1, 0)$ and $(-1, 0)$.

11. Find the maximum and minimum values of $f(x, y) = xy - y + x$ on the set D which is the interior and boundary of the closed triangular region with vertices $(0, 0)$,$(2, 0)$ and $(0, 3)$.

    Solution: First we check for local extrema, which must satisfy $f_x = y + 1 = 0$ and $f_y = x - 1 = 0$. Any solution would have to satisfy $y = -1$, and so could not possibly lie in the region D. Thus there are no local extrema, and so the minimum and maximum values must lie on the boundary. First we check $x = 0$. Here our function satisfies $f(0, y) = -y$ and so we get a maximum of 0 at $(0, 0)$ and a minimum of -3 at $(0, 3)$. Next we look at $y = 0$. Here the function satisfies $f(x, 0) = x$, and so we get a minimum of 0 at $(0, 0)$ and a maximum of 2 at $(2, 0)$. Finally we check the line segment connecting $(2, 0)$ and $(0, 3)$. The equation of the line is $y = 3 - \frac{3x}{2}$ and so the function satisfies $f(x) = \frac{-3x^2}{2} + \frac{9x}{2} + x - 3$. We check for relative extrema and get $f'(x) = -3x + \frac{9}{2} + 1 = 0 \implies x = \frac{11}{6}$. Thus the point $(\frac{11}{6}, \frac{1}{4})$ is a local extremum (on the line), and the value of the function at that point is $\frac{49}{24}$. There are no candidates left for the max and min, so the minimum value (attained at $(0, 3)$) is -3, and the maximum value (attained at $(\frac{11}{6}, \frac{1}{4})$) is $\frac{49}{24}$.