Midterm Exam 2

Math-M311 Spring 2011
March 10, 2011

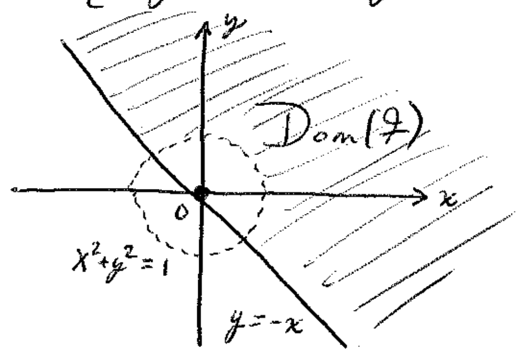
> Answer the questions in the spaces provided on the question sheets, being sure to provide full justification for your solutions. If you run out of room for an answer, continue on the back of the page. Your exam should have 5 pages. Please check to make sure your exam is complete.

Name: Solutions

1. (a) (25 points) Determine and sketch the domain of the function

$$f(x,y) = \sqrt{x+y}\ \ln(x^2+y^2-1).$$

$$\mathrm{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : x+y \geq 0,\ x^2+y^2 > 1\}$$



(b) Find the following limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{6x^2y^3}{\sqrt{x^2+2y^2}}.$$

Solution ① Use Polar coordinates $x = r\cos\theta, y = r\sin\theta$

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \frac{6x^2y^3}{\sqrt{x^2+2y^2}} = \lim_{r\to 0^+} \frac{6r^5\cos^2\theta\sin^3\theta}{r\sqrt{1+\sin^2\theta}} = 0.$$

Solution ② Use Squeeze Thm: $\dfrac{|x|}{\sqrt{x^2+2y^2}} = \dfrac{1}{\sqrt{1+2(y/x)^2}} \leq 1$

So, $0 \leq \left|\dfrac{6x^2y^3}{\sqrt{x^2+2y^2}}\right| = \left(\dfrac{|x|}{\sqrt{x^2+2y^2}}\right)\cdot 6|x||y|^3 \leq 6|x||y|^3 \to 0$ as $(x,y)\to(0,0)$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{6x^2y^3}{\sqrt{x^2+2y^2}} = 0.$$

(c) Determine the set of all points where the following function is continuous:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+3y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(Don't forget to explain what happens away from $(0,0)$.)

The fcns. xy and x^2+3y^2 are polynomials, and hence $f(x,y)$ is cts. so long as $x^2+3y^2 = 0$, i.e at $(0,0)$.

At $(0,0)$, have

$f(x,y) \to 0$ as $(x,y) \to (0,0)$ along x-axis

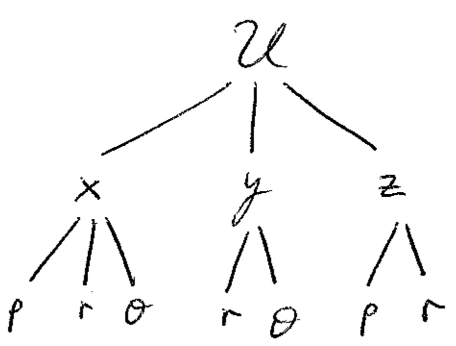
and

$f(x,y) \to \frac{1}{4}$ as $(x,y) \to (0,0)$ along line $y=x$

so that $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E. so f is not cts. at $(0,0)$.

$\therefore f$ is cts. at all $(x,y) \neq (0,0)$.

(d) If $u = 2x^3 + yz^2$, where $x = pr\cos(\theta)$, $y = r^2\sin(\theta)$ and $z = p + r$, find $\frac{\partial u}{\partial p}$ when $p = 2$, $r = 3$, $\theta = \frac{\pi}{2}$.



$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p}$$

$$= (6x^2)\cdot(r\cos\theta) + (2yz)\cdot(1)$$

When $(p, r, \theta) = (2, 3, \frac{\pi}{2})$, $(x,y,z) = (0,9,5)$

So that

$$\frac{\partial u}{\partial p}(2,3,\tfrac{\pi}{2}) = 2\cdot 9\cdot 5 = 90.$$

2. (25 points) Consider the function $f(x, y) = ye^{xy}$.

(a) Find the directional derivative of f at the point $(1, 2)$ in the direction of the vector $\vec{v} = \langle 3, -4 \rangle$.

$$f_x = y^2 e^{xy}, \quad f_y = e^{xy}(1+xy)$$

$$\Longrightarrow \nabla f(1,2) = \langle 4e^2, 3e^2 \rangle.$$

Since the unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$, the directional derivative is

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = 0.$$

(b) Find the maximum rate of change of f at the point $(1, 2)$, and determine the direction in which it occurs.

By a theorem in class, the maximum rate of change of f at $(1,2)$ is

$$|\nabla f(1,2)| = \sqrt{(4e^2)^2 + (3e^2)^2} = 5e^2$$

and it occurs in the direction of the vector

$$\nabla f(1,2) = \langle 4e^2, 3e^2 \rangle.$$

3. (25 points) Consider the function $f(x,y) = \sqrt{y + \cos^2(x)}$.

(a) Find an equation for the tangent plane to the surface $z = f(x,y)$ at the point $(0,0)$.

$$f_x = \frac{-\cos x \sin x}{\sqrt{y + \cos^2 x}}, \quad f_y = \frac{1}{2\sqrt{y+\cos^2 x}}$$

$$\Longrightarrow f_x(0,0) = 0, \quad f_y(0,0) = \tfrac{1}{2}.$$

So, we can use $\vec{n} = \langle 0, \tfrac{1}{2}, -1 \rangle$ for normal and hence tangent plane is

$$\tfrac{1}{2} y - (z-1) = 0.$$

(b) Use part (a) to approximate the value of $f(-0.05, 0.1) = \sqrt{0.1 + \cos^2(-0.05)}$.

By above, tangent plane at $(0,0)$ is $z = \tfrac{1}{2} y + 1$, so

$$f(-0.05, 0.1) \approx \tfrac{1}{2}(0.1) + 1 = 1.05.$$

(c) Show the surface $z = f(x,y)$ intersects the surface $2z^2 + (x+4)y - 2z\cos(x) = 0$ perpendicularly at the point $(0,0,1)$. (This means that the surfaces intersect at this point and that their tangent planes are perpendicular at this point of intersection.)

Setting $F(x,y,z) = 2z^2 + (x+4)y - 2z\cos x$, we see $F(0,0,1) = 0$ so the two surfaces indeed intersect at $(0,0,1)$. The normal to tangent plane of level surface $F(x,y,z) = 0$ at $(0,0,1)$ is

$$\nabla F(0,0,1) = \langle y + 2z\sin x,\ x+4,\ 4z - 2\cos x \rangle \Big|_{(x,y,z)=(0,0,1)}$$
$$= \langle 0, 4, 2 \rangle.$$

Since $\vec{n} \cdot \nabla F(0,0,1) = \langle 0, \tfrac{1}{2}, -1 \rangle \cdot \langle 0, 4, 2 \rangle = 0$, surfaces intersect perpendicularly at $(0,0,1)$.

4. (20 points) Find and classify the critical points of the function $f(x,y) = x^3 - 12xy + 8y^3$.

$f_x = 3x^2 - 12y$, $f_y = -12x + 24y^2$.

So, $f_x = 0 \Longrightarrow y = \frac{1}{4}x^2$ in which case

$$f_y = 12\left(-x + 2\cdot\tfrac{1}{16}x^4\right)$$
$$= 12x\left(-1 + \tfrac{1}{8}x^3\right) = 0$$

$\Longrightarrow$ $x = 0$ or $x = 2$.

Critical points are thus $(0,0)$ and $(2,1)$.

Now, $f_{xx} = 6x$, $f_{xy} = -12$, $f_{yy} = 48y$.

At $(0,0)$, $D(0,0) = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} < 0 \Longrightarrow$ $(0,0)$ is a saddle pt.

At $(2,1)$, $D(2,1) = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix} > 0$. Since $f_{xx}(2,1) > 0$

have that $(2,1)$ is a local min.