MAT 307: Combinatorics

Lecture 12: Extremal results on finite sets

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1 Largest antichains

Suppose we are given a family $\mathcal{F}$ of subsets of $[n]$. We call $\mathcal{F}$ an *antichain*, if there are no two sets $A, B \in \mathcal{F}$ such that $A \subset B$. For example, $\mathcal{F} = \{S \subseteq [n] : |S| = k\}$ is an antichain of size $\binom{n}{k}$. How large can an antichain be? The choice of $k = \lfloor n/2 \rfloor$ gives an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. In 1928, Emanuel Sperner proved that this is the largest possible antichain that we can have. In fact, we prove a slightly stronger statement.

Theorem 1 (Sperner's theorem)**.** *For any antichain $\mathcal{F} \subset 2^{[n]}$,*

$$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

Since $\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$ for any $A \subseteq [n]$, we conclude that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof. We present a very short proof due to Lubell. Consider a random permutation $\pi : [n] \to [n]$. We compute the probability of the event that a prefix of this permutation $\{\pi_1, \ldots, \pi_k\}$ is in $\mathcal{F}$ for some k. Note that this can happen only for one value of k, since otherwise $\mathcal{F}$ would not be an antichain.

For each particular set $A \in \mathcal{F}$, the probability that $A = \{\pi_1, \ldots, \pi_{|A|}\}$ is equal to $k!(n-k)!/n!$, corresponding to all possible orderings of A and $[n] \setminus A$. By the property of an antichain, these events for different sets $A \in \mathcal{F}$ are disjoint, and hence

$$\Pr[\exists A \in \mathcal{F}; A = \{\pi_1, \ldots, \pi_{|A|}\}] = \sum_{A \in \mathcal{F}} \Pr[A = \{\pi_1, \ldots, \pi_{|A|}\}] = \sum_{A \in \mathcal{F}} \frac{|A|!(n-|A|)!}{n!} = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.$$

The fact that any probability is at most 1 concludes the proof. □

This has the following application. We note that the theorem actually holds for arbitrary vectors and any ball of radius 1, but we stick to the 1-dimensional case for simplicity.

Theorem 2. *Let $a_1, a_2, \ldots, a_n$ be real numbers of absolute value $|a_i| \geq 1$.. Consider the 2^n linear combinations $\sum_{i=1}^n \epsilon_i a_i$, $\epsilon_i \in \{-1, +1\}$. Then the number of sums which are in any interval $(x-1, x+1)$ is at most $\binom{n}{\lfloor n/2 \rfloor}$.*

An interpretation of this theorem is that for any random walk on the real line, where the i-th step is either $+a_i$ or $-a_i$ at random, the probability that after n steps we end up in some fixed interval $(x-1, x+1)$ is at most $\binom{n}{\lfloor n/2 \rfloor}/2^n = O(1/\sqrt{n})$.

Proof. We can assume that $a_i \geq 1$. For $\epsilon \in \{-1,+1\}^n$, let $I = \{i \in [n] : \epsilon_i = +1\}$. If $I \subset I'$, and ϵ' corresponds to I', we have

$$\sum \epsilon'_i a_i - \sum \epsilon_i a_i = 2 \sum_{i \in I' \setminus I} a_i \geq 2|I' \setminus I|.$$

Therefore, if I is a proper subset of I' then only one of them can correspond to a sum inside $(x-1, x+1)$. Consequently, the sums inside $(x-1, x+1)$ correspond to an antichain and we can have at most $\binom{n}{\lfloor n/2 \rfloor}$ such sums. □

Theorem 3 (Bollobás, 1965)**.** *If $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are two sequences of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$, then*

$$\sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

Note that if $A_1, \ldots, A_m$ is an antichain on $[n]$ and we set $B_i = [n] \setminus A_i$, we get a system of sets satisfying the conditions above. Therefore this is a generalization of Sperner's theorem.

Proof. Suppose that $A_i, B_i \subseteq [n]$ for some n. Again, we consider a random permutation $\pi : [n] \to [n]$. Here we look at the event that there is some pair (A_i, B_i) such that $\pi(A_i) < \pi(B_i)$, in the sense that $\pi(a) < \pi(b)$ for all $a \in A_i, b \in B_i$. For each particular pair (A_i, B_i), the probability of this event is $|A_i|!|B_i|!/(|A_i| + |B_i|)!$.

On the other hand, suppose that $\pi(A_i) < \pi(B_i)$ and $\pi(A_j) < \pi(B_j)$. Hence, there are points x_i, x_j such that the two pairs are separated by x_i and x_j, respectively. Depending on the relative order of x_i, x_j, we get either $A_i \cap B_j = \emptyset$ or $A_j \cap B_i = \emptyset$, which contradicts our assumptions. Therefore, the events for different pairs (A_i, B_i) are disjoint. We conclude that

$$\Pr[\exists i; (A_i, B_i) \text{ are separated in } \pi] = \sum_{i=1}^{m} \frac{|A_i|!|B_i|!}{(|A_i| + |B_i|)!} = \sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

□

This theorem has an application in the following setting. For a collection of sets $\mathcal{F} \subseteq 2^X$, we call $T \subseteq X$ a *transversal* of $\mathcal{F}$, if $\forall A \in \mathcal{F}; A \cap T \neq \emptyset$. One question is, what is the smallest transversal for a given collection of sets $\mathcal{F}$. We denote the size of the smallest transversal by $\tau(\mathcal{F})$.

A set system $\mathcal{F}$ is called τ-critical, if removing any member of $\mathcal{F}$ decreases $\tau(\mathcal{F})$. An example of a τ-critical system is the collection $\mathcal{F} = \binom{[k+\ell]}{k}$ of all subsets of size k out of $k+\ell$ elements. The smallest transversal has size $\ell+1$, because any set of size $\ell+1$ intersects every member of $\mathcal{F}$, whereas no set of size ℓ is a transversal, since its complement is a member of $\mathcal{F}$. Moreover, removing any set $A \in \mathcal{F}$ decreases $\tau(\mathcal{F})$ to ℓ, because then $\bar{A}$ is a transversal of $\mathcal{F} \setminus \{A\}$. This is an example of a τ-critical system of size $\binom{k+\ell}{k}$, where $\tau(\mathcal{F}) = \ell+1$ and $\forall A \in \mathcal{F}; |A| = k$.

Observe that if $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ is τ-critical and $\tau(\mathcal{F}) = \ell+1$, then there is a transversal $B_i, |B_i| = \ell$ for each i, which intersects each $A_j, j \neq i$. However, B_i does not intersect A_i, otherwise it would also be a transversal of $\mathcal{F}$. Therefore, Theorem 3 implies the following.

Theorem 4. *Suppose $\mathcal{F}$ is a τ-critical system, where $\tau(\mathcal{F}) = \ell+1$ and each $A \in \mathcal{F}$ has size k. Then*

$$|\mathcal{F}| \leq \binom{k+\ell}{k}.$$

2 Intersecting families

Here we consider a different type of family of subsets. We call $\mathcal{F} \subseteq 2^{[n]}$ *intersecting*, if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$. The question what is the largest such family is quite easy: For any set A, we can take only one of A and $[n] \setminus A$. Conversely, we can take exactly one set from each pair like this - for example all the sets containing element 1. Hence, the largest intersecting family of subsets of $[n]$ has size exactly 2^{n-1}.

A more interesting question is, how large can be an intersecting family of sets of size k? We assume $k \leq n/2$, otherwise we can take all k-sets.

Theorem 5 (Erdős-Ko-Rado)**.** *For any $k \leq n/2$, the largest size of an intersecting family of subsets of $[n]$ of size k is $\binom{n-1}{k-1}$.*

Observe that an intersecting family of size $\binom{n-1}{k-1}$ can be constructed by taking all k-sets containing element 1. To prove the upper bound, we use an elegant argument of Katona. First, we prove the following lemma.

Lemma 1. *Consider a circle divided into n intervals by n points. Let $k \leq n/2$. Suppose we have "arcs" $A_1, \ldots, A_t$, each A_i containing k successive intervals around the circle, and each pair of arcs overlapping in at least one interval. Then $t \leq k$.*

Proof. No point x can be the endpoint of two arcs - then they are either the same arc, or two arcs starting from x in opposite directions, but then they do not share any interval.

Now fix an arc A_1. Every other arc must intersect A_1, hence it must start at one of the $k-1$ points inside A_1. Each such endpoint can have at most one arc. □

Now we proceed with the proof of Erdős-Ko-Rado theorem.

Proof. Let $\mathcal{F}$ be an intersecting family of sets of size k. Consider a random permutation $\pi : [n] \to [n]$. We consider each set $A \in \mathcal{F}$ mapped onto the circle as above, by associating $\pi(A)$ with the respective set of intervals on the circle. Let X be the number of sets $A \in \mathcal{F}$ which are mapped onto contiguous arcs $\pi(A)$ on the circle. For each set $A \in \mathcal{F}$, the probability that $\pi(A)$ is a contiguous arc is $nk!(n-k)!/n! = n/\binom{n}{k}$. Therefore,

$$\mathbf{E}[X] = \sum_{A \in \mathcal{F}} \Pr[\pi(A) \text{ is contiguous}] = \frac{n}{\binom{n}{k}} |\mathcal{F}|.$$

On the other hand, we know by our lemma that $\pi(A)$ can be contiguous for at most k sets at the same time, because $\mathcal{F}$ is an intersecting family. Therefore,

$$\mathbf{E}[X] \leq k.$$

From these two bounds, we obtain

$$|\mathcal{F}| \leq \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}.$$

□