Alon-Babai-Suzuki's Conjecture related to binary codes in non-modular version

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Abstract

Let $K = \{k_1, k_2, ..., k_r\}$ and $L = \{l_1, l_2, ..., l_s\}$ be sets of nonnegative integers. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ with $|F_i| \in K$ for each i and $|F_i \cap F_j| \in L$ for any $i \neq j$. Every subset F_e of $[n]$ can be represented by a binary code $\mathbf{a} = (a_1, a_2, ..., a_n)$ such that $a_i = 1$ if $i \in F_e$ and $a_i = 0$ if $i \notin F_e$. Alon-Babai-Sukuki made a conjecture in 1991 ([2]) in modular version. We prove Alon-Babai-Sukuki's Conjecture in non-modular version: For any K and L with $n \geq s + \max k_i$, $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

1 Introduction

In this paper, $\mathcal{F}$ stands for a family of subsets of $[n] = \{1, 2, ..., n\}$, $K = \{k_1, ..., k_r\}$ and $L = \{l_1, ..., l_s\}$ where $|F_i| \in K$ for all $F_i \in \mathcal{F}$, $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathcal{F}, i \neq j$. The variable x will stand as a short-hand for the n-dimensional vector variable $(x_1, x_2, ..., x_n)$. Also, since these variables will take the values only 0 and 1, all the polynomials we will work with will be reduced modulo the relation $x_i{}^2 = x_i$. We define the characteristic vector $v_i = (v_{i1}, v_{i2}, \cdots, v_{in})$ of F_i such that $v_{ij} = 1$ if $j \in F_i$ and $v_{ij} = 0$ if $j \notin F_i$. We will present some results in this paper that give upper bounds on the size of $\mathcal{F}$ under various conditions. Below is a list of related results by others.

Theorem 1. *(Ray-Chaudhuri and Wilson [1]) If $K = \{k\}$, and L is any set of nonnegative integers with $k > \max l_j$, then $|\mathcal{F}| \leq \binom{n}{s}$.*

Theorem 2. *(Alon, Babai, and Suzuki [2]) If K and L are two sets of nonnegative integers with $k_i > s - r$, for every i, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

Theorem 3. *(Snevily [3]) If K and L are any sets such that $\min k_i > \max l_j$, then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.*

Theorem 4. *(Snevily [7]) Let K and L be sets of nonnegative integers such that $\min k_i > \max l_j$. Then, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

Conjecture 5. *(Snevily [4]) For any K and L with $\min k_i > \max l_j$, $|\mathcal{F}| \leq \binom{n}{s}$.*

In the same paper in which he stated the above conjecture, Snevily mentions that it seems hard to prove the above bound and states the following weaker conjecture.

Conjecture 6. *(Snevily [4]) For any K and L with $\min k_i > \max l_j$, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.*

Hwang and Sheikh [8] proved the bound of Conjecture 6 when K is a consecutive set. The second theorem we prove is a special case of Conjecture 6 with the extra condition that $\bigcap_{i=1}^{m} F_i \neq \emptyset$. These two theorems are stated hereunder.

Theorem 7. *(Hwang and Sheikh [8]) Let $K = \{k_1, k_2, ..., k_r\}$ where $k_i = k_1 + i - 1$, $k_1 > s - r$, and $L = \{l_1, l_2, ..., l_s\}$. Let $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i| \notin L$, and $|F_i \cap F_j| \in L$ for any $i \neq j$. Then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.*

Theorem 8. *(Hwang and Sheikh [8]) Let $K = \{k_1, k_2, ..., k_r\}$, $L = \{l_1, l_2, ..., l_s\}$, and $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i \cap F_j| \in L$ for any $i \neq j$, and $k_i > s - r$. If $\bigcap_{i=1}^{m} F_i \neq \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.*

Theorem 9. *(Alon, Babai and Suzuki [2]) Let K and L be subsets of $\{0, 1, ..., p-1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ be a family of subsets of $[n]$ such that $|F_i|(\text{mod } p) \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j|(\text{mod } p) \in L$ for $i \neq j$. If $r(s - r + 1) \leq p - 1$, and $n \geq s + \max k_i$, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

Conjecture 10. *(Alon, Babai and Suzuki [2]) Let K and L be subsets of $\{0, 1, ..., p-1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ be a family of subsets of $[n]$ such that $|F_i|(\text{mod } p) \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j|(\text{mod } p) \in L$ for $i \neq j$. If $n \geq s + \max k_i$, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

In [2], Alon, Babai, and Suzuki proved their conjectured bound under the extra conditions that $r(s - r + 1) \leq p - 1$ and $n \geq s + \max k_i$. Qian and Ray-Chaudhuri ([5]) proved that if $n > 2s - r$ instead of $n \geq s + \max k_i$, then the above bound holds.

We prove a Alon-Babai-Suzuki's conjecture in non-modular version.

Theorem 11. *Let $K = \{k_1, k_2, ..., k_r\}$, $L = \{l_1, l_2, ..., l_s\}$ be two sets of nonnegative integers and $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i \cap F_j| \in L$ for any $i \neq j$, and $n \geq s + \max k_i$. then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

2   Proof of Theorem

Proof of Theorem 11   For each $F_i \in \mathcal{F}$, consider the polynomial

$$f_i(x) = \prod_{\substack{j \\ l_j < |F_i|}} (v_i \cdot x - (k_i - l_j)),$$

where v_i is the characteristic vector of F_i and v_i^* is the characteristic vector of $F_i^* = F_i - \{1\}$. Let $\overline{v_i}$ be the characteristic vector of F_i^c, and $\overline{v_i}^*$ be the characteristic vector of $(F_i^c)^*$.

We order $\{F_i\}$ by size of F_i, that is, $|F_j| \leq |F_k|$ if $j < k$. We substitute the characteristic vector $\overline{v_i}$ of F_i^c by order of size of F_i. Clearly, $f_i(\overline{v_i}) \neq 0$ for $1 \leq i \leq r$ and $f_i(\overline{v_j}) = 0$ for $1 \leq j < i \leq r$. Assume

$$\sum_i \alpha_i f_i(x) = 0.$$

We prove that $\{f_i(x)\}$ is linearly independent. Assume that this is false. Let i_0 be the smallest index such that $\alpha_{i_0} \neq 0$. We substitute $\overline{v_{i_0}}$ into the above equation. Then we get $\alpha_{i_0} f_{i_0}(\overline{v_{i_0}}) = 0$. We get a contradiction. So $\{f_i(x)\}$ is linearly independent. Let $\mathcal{E} = \{E_1, ..., E_e\}$ be the family of subsets of $[n]$ with size at most $s - r$, which is ordered by size, that is, $|E_i| \leq |E_j|$ if $i < j$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$. Let u_i denote the characteristic vector of E_i. We define the multilinear polynomial g_i in n variables for each E_i:

$$g_i(x) = \prod_{l=1}^{r} (\sum_{t=1}^{n} x_t - (n - k_l)) \prod_{j \in E_i} x_j.$$

We prove that $\{g_i(x)\}$ is linearly independent. Assume that

$$\sum_i \beta_i g_i(x) = 0.$$

Choose the smallest size of E_i. Let u_i be the characteristic vector of E_i. We substitute u_i into the above equation. We know that $g_i(u_i) \neq 0$ and $g_j(u_i) = 0$ for any $i < j$. Since $n \geq s + \max k_i$, we get $\beta_i = 0$. If we follow the same process, then the family $\{g_i(x)\}$ is

linearly independent. Next, we prove that $\{f_i(x), g_i(x)\}$ is linearly independent. Now, assume that

$$\sum_i \alpha_i f_i(x) + \sum_i \beta_i g_i(x) = 0.$$

Let F_1 be the smallest size of F_i. We substitute the characteristic vector $\overline{v_1}$ of F_1^c into the above equation. Since $|F_i^c| = n - k_l$, $g_i(\overline{v_1}) = 0$ for all i. We only get $\alpha_1 f_1(\overline{v_1}) = 0$. So $\alpha_1 = 0$. By the same way, choose the smallest size from $\{F_i\}$ after deleting F_1. We do the same process. We also can get $\alpha_2 = 0$. By the same process, we prove that all $\alpha_i = 0$. We prove that $\{f_i(x), g_i(x)\}$ is linearly independent.

Any polynomial in the set $\{f_i(x), g_i(x)\}$ can be represented by a linear combination of multilinear monomials of degree $\leq s$. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s} \binom{n}{i}$. We found $|\mathcal{F}| + \sum_{i=0}^{s-r} \binom{n}{i}$ linearly independent polynomials with degree at most s. So $|\mathcal{F}| + \sum_{i=0}^{s-r} \binom{n}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$. Thus $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. $\blacksquare$

Acknowledgement

References

[1] D. K. Ray-Chaudhuri and R. M. Wilson, On t-designs, *Osaka. J. Math*, **12** (1975), 737-744.

[2] N. Alon, L. Babai, and H. Suzuki, Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems, *J. Comb. Theory Ser.A*, **58** (1991), 165-180.

[3] H. S. Snevily, On generalizations of the deBruijn-Erdős Theorem, *J. Combin. Theory Ser. A*, **68** (1994), 232-238.

[4] H. S. Snevily, A generalization of the Ray Chaudhuri-Wilson Theorem, *J. Combin. Designs*, **3** (1995), 349-352.

[5] Jin Qian and D. K. Ray-Chaudhuri, On mod-p Alon-Babai-Suzuki inequality, *J. Algebraic Combin.*, **12** (2000), 85-93.

[6] Jin Qian and D. K. Ray-Chaudhuri, Extremal case of Frankl-Ray-Chaudhuri-Wilson inequality, Special issue on design combinatorics: in honor of S. S. Shrikhande, *J. Statist. Plann. Inference*, **95** (2001), 293-306.

[7] H. S. Snevily, A sharp bound for the number of sets that pairwise intersect at k positive values, *Combinatorica*, **23** (2003), 521-533.

[8] K-W. Hwang and N. Sheikh, Intersection families and Snevily's conjecture, *European Journal of Combinatorics*, **28** (2007), 843-847.

[9] W. Cao, K-W. Hwang, and D. West, Improved Bounds on Families Under k-wise Set-Intersection Constraints, *Graphs and Combinatorics* , **23** (2007), 381-386.

[10] Z. Furedi, K-W. Hwang, and P. Weichsel, A proof and generalizations of the Erdos-Ko-Rado theorem using the method of linearly independent polynomials, *Topics in Discrete Mathematics*, **26** (2006), 215-224.