M463 Extra Credit 1

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Find $\left(\frac{1}{2}\right)!$, i.e., the factorial of one-half.

Solution: Using the Gamma function we know that for any $r \in \mathbb{R}$

$$\Gamma(r+1) = r!, \qquad \text{where} \quad \Gamma(r+1) = \int_0^\infty e^{-t} t^r dt$$

First note an important property of this function, i.e.:

$$\Gamma(r+1) = r\Gamma(r), \quad \text{for any } r$$

Proof: Using integration by parts:

$$\Gamma(r+1) = \int_0^\infty t^r e^{-t} dt = \left[-e^{-t}t^r\right]_0^\infty + \int_0^\infty e^{-t} r t^{r-1} dt = r\int_0^\infty e^{-t} t^{r-1} dt = r\Gamma(r) \qquad \square$$

Using this property, it suffices to compute $\Gamma\left(\frac{1}{2}\right)$ to obtain $\left(\frac{1}{2}\right)!$ since $\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$. Hence,

$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$ change $t = u^2 \Longrightarrow dt = 2udu$. Note that $u \to 0$ as $t \to 0$ and $u \to \infty$ as $t \to \infty$. Thus:

$$\int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \int_0^\infty e^{-u^2} u^{2^{-\frac{1}{2}}} 2udu = 2\int_0^\infty e^{-u^2} u^{-1} udu = 2\int_0^\infty e^{-u^2} du$$

This new integral can be solved by doble integration as follows:

$$\left(2\int_0^\infty e^{-u^2} du\right)^2 = 4\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy \quad \text{switch to polar coordinates:}$$

$$4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta = 2\pi \int_0^\infty e^{-r^2} r dr \quad \text{change } r^2 = w \Longrightarrow 2rdr = dw$$

$$= 2\pi \int_0^\infty \frac{e^{-w}}{2} dw = \pi \left[-e^{-w}\right]_0^\infty = \pi(0+1) = \pi \qquad \text{Therefore,} \quad 2\int_0^\infty e^{-u^2} du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Using the property proved above, we conclude that:

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \boxed{\frac{1}{2}\sqrt{\pi}}$$