M413 - Fall 2013 - Enrique Areyan - Test 2

Problem 1: Part (b) Let $g(x) = f(x) - f(1+x)$. ; $x \in [0,1]$.

Now consider the following cases:

If $f(1) = f(0) = f(2)$ then:

$$g(0) = f(0) - f(1) = f(1) - f(1) = 0 \Rightarrow f(0) = f(1)$$

So $x = 0$ is s.t. $x \in [0,2]$ and $f(x) = f(0) = f(1) = f(x+1$

Otherwise: Suppose $f(1) \neq f(0) \Rightarrow f(1) \neq f(2)$, since $f(0)=f($

Note that f is continuous on $[0,2]$ and since g is the differen

between two continuous function we can conclude that g is continuou

therefore, we can use the Intermediate Value Property on g

as follows:

$$g(0) = f(0) - f(1) = f(2) - f(1) \quad (\text{since } f(0) = f(2)).$$

$$g(1) = f(1) - f(2)$$

So, Consider the cases:

(i) $f(1) > f(2)$. then $f(2) - f(1) < 0$ and $f(1) - f(2) > 0$

But then, by our definition of g:

$g(0) < 0 < g(1)$. By I.V.P, there exists $x \in [0,1]$ s.t.

$g(x) = 0 \Rightarrow g(x) = 0 = f(x) - f(x+1) \Rightarrow f(x) = f(x+1)$, so

x is the value we wanted.

(ii) $f(1) < f(2)$. then $f(2) - f(1) > 0$ and $f(1) - f(2) < 0$

$g(0) > 0 > g(1)$. Likewise, the result follows, i.e., there

exists $x \in [0,1]$ s.t $g(x) = 0 \Rightarrow g(x) = 0 = f(x) - f(x+1) \Rightarrow f(x) = f(x+$

In either case let $x_1 = x$ and $x_2 = 1+x$. Note that since $x \in [0,$

$x_1 \in [0,1]$ and $x_2 \in [0,2]$.

✓

Problem 3 : Part (b).

onsider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test,
.e., by Leibnitz result : If $c_n = \frac{(-1)^n}{n}$, then (a) $|c_1| \geq |c_2| \geq \cdots$
b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ and (c) $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$.

ow, If we let $x_n = \frac{(-1)^n}{n} = y_n$, then $x_n y_n = \frac{(-1)^{2n}}{n} = \frac{1}{n^2}$, for all n,
nd then $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is the harmonic series and
e know it diverges. (theorem 3.28 with $p = 1$).

f one of the series is absolutely convergent, say x_n, then
$\sum_{n=1}^{\infty} |x_n|$ converges then $\sum_{n=1}^{\infty} x_n$ converges and we will have that:

(a) $\sum_{n=1}^{\infty} x_n = A$

(b) $\sum_{n=1}^{\infty} y_n = B$

hen we know by theorem NO 3.50 that the Cauchy product
ill converge and moreover, it will converge to the right place,
.e., If $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ than $\sum_{n=0}^{\infty} c_n = AB$.

M413- Fall 2013 - Enrique Areyan - Test 2 ②

Problem 5 : Part (b). YES, f is necessarily continuous. NO

Let $p \in X$. We say that f is continuous at ~~f~~ p If: (Note domain(f) = X)

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in X$: If $d(x,p) < \delta$ then $d(f(x), f(p)) < \varepsilon$.

Now, f is continuous if it is continuous for every point in X.

So, Let $p \in X$. By hypothesis there exists a sequence $\{p_n\} \subset X$ s.t. $p_n \to p$ and $f(p_n) \to f(p)$. Now, assume for a contradiction that f is not continuous on X. this means that:

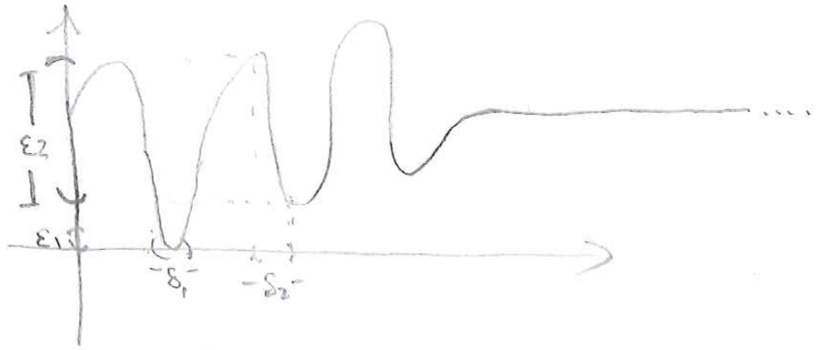
$\exists p \in X : \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in X : d(x,p) < \delta$ but $d(f(x), f(p)) > \varepsilon$.

Now, pick such a p and ε. Since $\delta > 0$ can be arbitrary, pick $\delta_n = \frac{1}{n}$

You will obtain a sequence $\{x_n\}$ s.t. $d(x_n, p) < \frac{1}{n}$ but $d(f(x_n), f(p)) > \varepsilon$. In particular, this means that we have obtained a point $p \in X$ such that $\{x_n\} \to p$ but $f(x_n)$ does not converge to $f(p)$ since the distance between $f(x_n)$ and $f(p)$ can be made arbitrarily small. therefore, we reach a contradiction.

what contradiction?

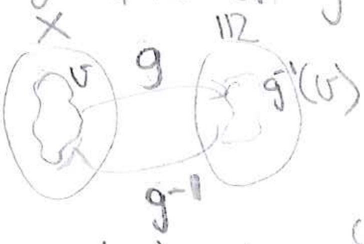
Problem 2: Part (a). By pictures:



the function might oscillate at the beginning but the Tail must necessarily converge since f has a horizontal asymptote at ∞ $\lim_{x\to\infty} f(x)$ exists.

herefore, there are finitely many places before the tail of the unction for which we might need different δ's to be within a given ε. However, we can take the minimum of all such δ's and we now it will work for all given ε. If it works for all ε efore the tail it will clearly work for the tail because f onverges as $x \to \infty$. Now, we are guaranteed the existence of uch δ because f is continuous therefore f is uniformly ontinuous, just pick the $\min\{\delta_1, \delta_2, \dots, \delta_n\}$ for all finite scillations of f before the tail. Idea is there, proof is incomplete.

oblem 4: Part (b). We want to prove that g is continuous sing the open set characterization, Let us prove that for every open set $U \subset \mathbb{R} \Rightarrow g^{-1}(U)$ is open in X. we want to show that if $x \in g^{-1}(U)$ then x is interior to $^{-1}(U)$. By definition $g^{-1}(U) = \{x \in X : g(x) \in U\} = \{x \in X : d(x, f(x)) \in U\}$.



Interior to $g^{-1}(U)$ means that there exist $N_r(x) \subset g^{-1}(U)$. Moreover, by definition, $_r(x) = \{y \in X : d(x,y) < r\}$. By hypothesis f is continuous. So in particular t is continuous at x. Let $\varepsilon > 0$, we can find $\delta > 0$ s.t $d(x,y) < \delta$ then $(f(x), f(y)) < \varepsilon$. In particular, $d(x, f(x)) < \delta$ then $d(f(x), f(f(x))) < \varepsilon$.

Problem 4: Part (b) (cont.)

therefore, Given $x \in g^{-1}(U)$, we can always find r small enough so that $N_r(x) = \{y \in X : d(x,y) < r\}$ is totally contained in $g^{-1}(U)$. Hence g is continuous.

An alternative proof would be directly from the definition.

Want to prove:

$\forall p \in X : \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in X : d(x,p) < \delta \Rightarrow d(g(x), g(p)) < \varepsilon$.

Fix $p \in X$. Let $\varepsilon > 0$. Pick $\delta > 0$ to be s.t. $d(g(x), p) \leq \frac{\varepsilon}{2}$ and $d(p, g(p)) \leq \frac{\varepsilon}{2}$ which you can do because f is continuous.

then:

$$d(g(x), g(p)) \overset{?}{=} d(\underbrace{d(x, f(x))}, \underbrace{d(p, f(p))})$$

d is a distance on X, $d(x, f(x)) \in \mathbb{R}$.

$$\leq d(g(x), p) + d(p, g(p))$$
$$= d(d(x, f(x)), p) + d(p, d(p, f(p)))$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow d(g(x), g(p)) < \varepsilon$, but since p was arbitrarily in X choosen, we can conclude that g is continuous in X