S413 - FIRST TEST

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If a problem has two parts, do ONLY one; if you do both you will get the lower grade. And, write legibly. I will only grade what I can read.

Problem 1. Part (a) Let $\mathcal{P}$ be the set of all real polynomials with rational coefficients. That is, $\mathcal{P} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{Q}, n = 1, 2, \ldots\}$. Prove that $\mathcal{P}$ is countable.

Part (b) Let $\mathcal{F}(\mathbb{N})$ be the collection of finite subsets of $\mathbb{N}$. That is, $F(\mathbb{N}) = \{A : A \text{ is finite}, A \subset \mathbb{N}\}$. Prove that $\mathcal{F}(\mathbb{N})$ is countable.

Problem 2. Let X be a metric space with metric d. Prove that

$$d_1(p,q) = \min\{d(p,q), 1\}$$

defines a metric on X that is equivalent to d. In other words,

$$\lim_n d(p_n, p) = 0 \text{ if and only if } \lim_n d_1(p_n, p) = 0\,.$$

Problem 3. Part (a) Let X be a metric space with metric d, and $\{p_n\}$ a sequence in X. Suppose that $\lim_n p_{2n} = p = \lim_n p_{2n+1}$. Prove that $\lim_n p_n = p$.

Part (b) Let X be a metric space with metric d, $A \subset X$, and $p \in X$. Prove that $d(p, A) = 0$ if and only if $p \in \overline{A}$.

Recall that $d(p, A) = \inf\{d(p,q) : q \in A\}$.

Problem 4. Part (a) Give an example of real sequences $\{p_n\}$ and $\{q_n\}$ such that $\{p_n\}$ is bounded and $\{q_n\}$ is convergent, but $\{p_n + q_n\}$ and $\{p_n q_n\}$ are divergent.

Part (b) Let X be the set of real numbers $x \in [0,1]$ such that 4 does not appear in the decimal expansion of x. That is, $X = \{x \in [0,1] : x = .x_1 x_2 x_3 \ldots, x_n \neq 4 \text{ for all } n\}$. Is X dense in $[0,1]$?

Problem 5. Part (a) Let $K_1, \ldots, K_n$ be a finite collection of compact subsets of a metric space X with metric d. Prove that $K_1 \cup K_2 \cup \cdots \cup K_n$ is compact. Show (by example) that this result does not extend to infinite unions.

Part (b) Let $A_1, \ldots, A_n$ be a finite collection of subsets of a metric space X with metric d, and $A = \bigcap_{k=1}^{n} A_k$. Prove that $\mathrm{int}(A) = \bigcap_{k=1}^{n} \mathrm{int}(A_k)$. Show (by example) that this result does not extend to infinite intersections.